Polymer Parametrised Field Theory

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Abstract

Free scalar field theory on 2 dimensional flat spacetime, cast in diffeomorphism invariant guise by treating the inertial coordinates of the spacetime as dynamical variables, is quantized using LQG type 'polymer' representations for the matter field and the inertial variables. The quantum constraints are solved via group averaging techniques and, analogous to the case of spatial geometry in LQG, the smooth (flat) spacetime geometry is replaced by a discrete quantum structure. An overcomplete set of Dirac observables, consisting of (a) (exponentials of) the standard free scalar field creation- annihilation modes and (b) canonical transformations corresponding to conformal isometries, are represented as operators on the physical Hilbert space. None of these constructions suffer from any of the 'triangulation' dependent choices which arise in treatments of LQG. In contrast to the standard Fock quantization, the non- Fock nature of the representation ensures that the algebra of conformal isometries as well as that of spacetime diffeomorphisms are represented in an anomaly free manner. Semiclassical states can be analysed at the gauge invariant level. It is shown that 'physical weaves' necessarily underly such states and that such states display semiclassicality with respect to, at most, a countable subset of the (uncountably large) set of observables of type (a). The model thus offers a fertile testing ground for proposed definitions of quantum dynamics as well as semiclassical states in LQG.

1 Introduction

This work is devoted to an application of canonical Loop Quantum Gravity (LQG) techniques to the quantization of a generally covariant, field theoretic toy model which goes by the name of Parametrised Field Theory

(PFT). PFT is just free field theory on flat spacetime, cast in a diffeomorphism invariant disguise. It offers an elegant description of free scalar field evolution on *arbitrary* (and in general curved) foliations of the background spacetime by treating the 'embedding variables' which describe the foliation as dynamical variables to be varied in the action in addition to the scalar field. Specifically, let $X^A = (T, X)$ denote inertial coordinates on 2 dimensional flat spacetime. In PFT, X^A are parametrized by a new set of arbitrary coordinates $x^\alpha = (t, x)$ such that for fixed t, the embedding variables $X^A(t, x)$ define a spacelike Cauchy slice of flat spacetime. General covariance of PFT ensues from the arbitrary choice of x^α and implies that in its canonical description, evolution from one slice of an arbitrary foliation to another is generated by constraints. While 2 dimensional PFT has been quantized in a Fock representation for the matter fields in References [1, 2], here we are interested in the construction of an LQG type representation for both the embedding as well as the matter fields, along the lines of Reference [3]. The usefulness of this exercise for canonical LQG can only be gauged in the context of the current status of the field, a brief discussion of which we now turn to.

LQG is a non- perturbative approach to quantum gravity which, in its canonical version, attempts to construct a Dirac quantization of a Hamiltonian description of gravity in terms of a spatial $SU(2)$ connection. and its conjugate electric field. The strength of this approach is that it constitutes, for the most part, an extremely conservative development and application of canonical quantization techniques to gravity (see for e.g. the reviews [4, 5, 6, 7]). This conservative union of the principles of quantum mechanics with those of classical gravity has yielded many beautiful results such as a satisfactory treatment of spatial diffeomorphisms [8, 9], discrete spatial geometry [10, 11, 12], a calculation of black hole entropy [13, 14] and a uniqueness theorem for its underlying representation [15, 16]. However, a necessity for radical ideas has arisen in the treatment of quantum dynamics [17, 18] as well as in that of semiclassical issues [19, 20, 21].

The key obstruction to a completely conservative treatment stems from the fact that in LQG only certain non- local functionals of the connection, namely the holonomies around spatial loops, can be promoted to quantum operators rather than the connection itself. [1] As a result, all questions of interest (including that of the quantum dynamics defined by the Hamiltonian constraint which is a *local* function of the connection and triad,) need to be phrased in terms of holonomy operators. Since holonomy operators associated with close by loops have actions unrelated by any sort of continuity, this leads to a situation where a *choice* of a subset of the (uncountable) set

[1] The reason for this is the lack of regularity in the action of the holonomy operators: while, classically, the connection at a point can be obtained from the holonomy of a loop containing the point in the limit that the loop is infinitesimally small, the limit of the corresponding operators does not exist in the LQG representation.

of all holonomy operators (or equivalently, the spatial loops labelling them) becomes necessary. We shall loosely refer to such choices as "triangulation" choices since, often, the family of loops is chosen to lie on some set of triangulations of the spatial manifold. Since there seems to be no natural choice independent of the intuition of the researcher, this leads to proposals which may be seen as radical or ad- hoc depending on ones taste.

In order to test these proposals it is necessary to have a 'perfect' toy model in which an LQG type of quantization can be constructed which is free from any triangulation ambiguities. What is needed is a generally covariant, *field theoretic* (with an infinite number of true degrees of freedom, since many of the difficulties can be traced to the field theoretic nature of gravity) system in which all steps of an LQG type quantization procedure can be carried out in a triangulation independent manner. As we show in this work, just such a model is provided by 2 dimensional parametrised field theory on $S^1 \times R$. Specifically, we construct, in a triangulation independent manner: an appropriate kinematic 'holonomy' algebra and its LQG type 'polymer' representation on a kinematic Hilbert space $\mathcal{H}_{kin}$, a representation on $\mathcal{H}_{kin}$ of both (the finite transformations generated by) the constraints and an over- complete set of gauge invariant observables, the group averaging map [22, 9] and the physical state space $\mathcal{H}_{phys}$ which naturally inherits a representation of the Dirac observables from that on $\mathcal{H}_{kin}$.

The above quantization of PFT offers an arena in which proposals for quantum dynamics developed for LQG may be tested against the manifestly triangulation/regularization free group averaging techniques used in this work. Further, semiclassical issues can be examined at the physical state level since both $\mathcal{H}_{phys}$ and representation of an overcomplete set of Dirac observables thereon, are available. This is in contrast to LQG wherein most current proposals are defined on $\mathcal{H}_{kin}$ with the hope that they may still be useful at the physical state level. Again, since the quantization here admits a representation of Dirac observables on $\mathcal{H}_{kin}$ as well as $\mathcal{H}_{phys}$, it offers a useful testing ground for proposed constructions of semiclassical states in LQG. Finally, since PFT also admits the usual Fock space quantization of the scalar field [1, 2], this can be compared with the "polymer" quantization presented here. This comparison is useful for similar 'graviton from LQG' issues [23] in canonical LQG.

The layout of the paper is as follows. Section 2 contains a brief review of classical PFT on $S^1 \times R$. Details may be found in [24]. In section 3, $\mathcal{H}_{kin}$ is constructed as the tensor product of Hilbert spaces for the matter and embedding sectors, each of which supports a polymer representation of suitably defined LQG- type operators. It is shown that $\mathcal{H}_{kin}$ also supports a unitary representation of the finite canonical transformations generated by the constraints. In section 4 an overcomplete set of gauge invariant (Dirac) observables corresponding to (a) exponentials of the standard mode functions of the free scalar field on flat spacetime and (b) conformal isome-

tries, are promoted to operators on $\mathcal{H}_{kin}$. These operators commute with those corresponding to finite gauge transformations. In section 5, the physical state space, $\mathcal{H}_{phys}$, is constructed through group averaging techniques [22, 9]. Ambiguities in the group averaging map are systematically reduced by requiring commutativity with the Dirac observables and superselection sectors are described, each of which provide a cyclic, *non- separable* representation of the algebra generated by the gauge invariant operators of section 4. Section 6 is devoted to a preliminary discussion of semiclassical issues. It is shown that, at most, only a countable subset of the overcomplete (and uncountable) set of Dirac observables of type (a) can be approximated by semiclassical states in $\mathcal{H}_{phys}$. Further, it is shown that any such state must be characterized by a suitably defined "physical" weave. Two issues (connected with the S^1 spatial topology and the treatment of zero modes) are addressed in section 7. Section 8 contains a discussion of our results as well as of open issues.

In the interests of brevity, we shall refrain from providing detailed proofs where such proofs are straightforward. Some Lemmas are proved in the Appendices A and B. The dimensions of various quantities and our choice of units are displayed in Appendix C.

2 Classical PFT on $S^1 \times R$.

We provide a brief review of classical 2 dimensional PFT. In sections 2.1 and 2.2 we shall implicitly assume that the spatial topology is that of a circle. The consequences of this non- trivial spatial topology on the formalism will be made explicit in section 2.3.

2.1 The Action for PFT.

The action for a free scalar field f on a fixed flat 2 dimensional spacetime in terms of global inertial coordinates $X^A,\ A = 0, 1$ is

$$S_0[f] = -\frac{1}{2}\int d^2X \eta^{AB}\partial_A f \partial_B f, \tag{1}$$

where the Minkowski metric in inertial coordinates, η^{AB}, is diagonal with entries $(-1, 1)$. If instead, we use coordinates x^α , $\alpha = 0, 1$ (so that X^A are 'parameterized' by x^α, $X^A = X^A(x)$), we have

$$S_0[f] = -\frac{1}{2}\int d^2x \sqrt{\eta}\eta^{\alpha\beta}\partial_\alpha f \partial_\beta f, \tag{2}$$

where $\eta_{\alpha\beta} = \eta_{AB}\partial_\alpha X^A \partial_\beta X^B$ and η denotes the determinant of $\eta_{\alpha\beta}$. The action for PFT is obtained by considering the right hand side of (2) as a functional, not only of ϕ, but also of $X^A(x)$ i.e. $X^A(x)$ are considered as

2 new scalar fields to be varied in the action ($\eta_{\alpha\beta}$ is a function of $X^A(x)$). Thus

$$S_{PFT}[f, X^A] = -\frac{1}{2}\int d^2x \sqrt{\eta(X)}\eta^{\alpha\beta}(X)\partial_\alpha f \partial_\beta f. \tag{3}$$

Note that S_{PFT} is a diffeomorphism invariant functional of the scalar fields $f(x), X^A(x)$. Variation of f yields the equation of motion $\partial_\alpha(\sqrt{\eta}\eta^{\alpha\beta}\partial_\beta f) = 0$, which is just the flat spacetime equation $\eta^{AB}\partial_A\partial_B f = 0$ written in the coordinates x^α. On varying X^A, one obtains equations which are satisfied if $\eta^{AB}\partial_A\partial_B f = 0$. This implies that $X^A(x)$ are undetermined functions (subject to the condition that determinant of $\partial_\alpha X^A$ is non- vanishing). This 2 functions- worth of gauge is a reflection of the 2 dimensional diffeomorphism invariance of S_{PFT}. Clearly the dynamical content of S_{PFT} is the same as that of S_0; it is only that the diffeomorphism invariance of S_{PFT} naturally allows a description of the standard free field dynamics dictated by S_0 on *arbitrary* foliations of the fixed flat spacetime.

2.2 Hamiltonian Formulation of PFT.

In the previous subsection, $X^A(x)$ had a dual interpretation - one as dynamical variables to be varied in the action, and the other as inertial coordinates on a flat spacetime. In what follows we shall freely go between these two interpretations.

We set $x^0 = t$ and $\{x^\alpha\} = \{t, x\}$. We restrict attention to $X^A(x)$ such that for any fixed t, $X^A(t, x^a)$ describe an embedded spacelike hypersurface in the 2 dimensional flat spacetime (it is for this reason that $X^A(x)$ are called embedding variables in the literature). This means that, for fixed t, the functions $X^A(x)$ must be such that the symmetric form q_{ab} defined by

$$q_{ab}(x) := \eta_{AB}\frac{\partial X^A(x)}{\partial x}\frac{\partial X^B(x)}{\partial x} \tag{4}$$

is an 1 dimensional Riemannian metric. This follows from the fact that $q_{ab}(x)$ is the induced metric on the hypersurface in the flat spacetime defined by $X^A(x)$ at fixed t.

A 1+1 decomposition of S_{PFT} with respect to the time 't', leads to its Hamiltonian form:

$$S_{PFT}[f, X^A; \pi, \Pi_A; N^A] = \int dt \int d^n x(\Pi_A \dot{X}^A + \pi_f \dot{f} - N^A H_A). \tag{5}$$

Here π_f is the momentum conjugate to the scalar field f, Π_A are the momenta conjugate to the embedding variables X^A, N^A are Lagrange multipliers for the first class constraints H_A. It turns out that the motions on phase space generated by the 'smeared' constraints, $\int d^n x(N^A H_A)$ correspond to scalar field evolution along arbitrary foliations of the flat spacetime, each choice of foliation being in correspondence with a choice of multipliers N^A.

Since the constraints are first class they also generate gauge transformations, and as in General Relativity, the notions of gauge and evolution are intertwined.

Since free scalar field theory in 2 dimensions finds its simplest expression in terms of left and right movers, it is useful to make a point canonical transformation to light cone embedding variables $X^{\pm}(x) := T(x) \pm X(x)$ (here we have set $X^0 = T, X^1 = X$). Denoting the conjugate embedding momenta by $\Pi^{\pm}(x)$, and setting $H_{\pm} = H_0 \pm H_1$, the action takes the form

$$S \;=\; \int dt \int dx \; [\; \pi_f \dot{f} \;\; + \Pi_+ \dot{X}^+ \;\; + \;\; \Pi_- \dot{X}^- \;\; - \;\; N^+ H_+ \;\; - \;\; N^- H_- \;]. \quad (6)$$

where $N^{\pm}$ are the new Lagrange multipliers appropriate to $H_{\pm}$. Explicitly, the constraints $H_{\pm}$ are given by

$$H_{\pm}(x) \;=\; [\; \Pi^{\pm}(x) X^{\pm '}(x) \;\pm\; \frac{1}{4}(\pi_f \;\pm f')(x)(\pi_f \;\pm f')(x) \;]. \quad (7)$$

Note that while $X^{\pm}(x), f(x)$ transform as scalars under spatial coordinate transformations, $\Pi_{\pm}, \pi_f, N^{\pm}$ transform as scalar densities (or equivalently as spatial vector fields).

The Poisson brackets between various fields are given by,

$$\begin{aligned} \{f(x), \pi_f(x')\} \;&=\; \delta(x, x'), \\ \{X^{\pm}(x), \Pi_{\pm}(x')\} \;&=\; \delta(x, x'), \end{aligned} \quad (8)$$

and the remaining brackets are zero. Here $\delta(x, x')$ is the delta-function on S^1.

To complete the transition to variables closely related to the left and right movers of free scalar field theory [24], we perform a canonical transformation on the matter variables. $(f, \pi_f) \;\rightarrow (Y^+, Y^-)$. Here $Y^{\pm}(x) \;=\; \pi_f(x) \;\pm\; f'(x)$ (strictly speaking this transformation is not invertible when the spatial topology is S^1 due to the existence of zero modes; we shall return to this issue in section 3). The Poisson brackets between the scalar densities, $Y^{\pm}$, are given by,

$$\begin{aligned} \{Y^{\pm}(x), Y^{\pm}(x')\} \;&=\; \pm[\partial_x \delta(x, x') \;-\; \partial_{x'} \delta(x', x)] \\ \{Y^{\mp}(x), Y^{\pm}(x')\} \;&=\; 0. \end{aligned} \quad (9)$$

The constraints are now

$$H^{\pm}(x) \;=\; [\; \Pi_{\pm}(x) X^{\pm '}(x) \;\pm\; \frac{1}{4} Y^{\pm}(x)^2 \;]. \quad (10)$$

and the constraint algebra is

$$\begin{aligned} \{H_{\pm}[N^{\pm}], H_{\pm}[M^{\pm}]\} \;&=\; H_{\pm}[\mathcal{L}_{N^{\pm}} M^{\pm}] \\ \{H_{\pm}[N^{\pm}], H_{\mp}[M^{\mp}]\} \;&=\; 0 \end{aligned} \quad (11)$$

Here $\mathcal{L}_N$ denotes the Lie derivative with respect to the 1 dimensional spatial vector field with component $N(x)$ in the coordinate system 'x'. The action of the constraints on the phase space variables can be expressed as follows. Let $\Phi^{\pm} = (Y^{\pm}, \Pi_{\pm})$, we have

$$\begin{aligned} \{\Phi^{\pm}(x),\ H_{\pm}[N^{\pm}]\ \} &= \mathcal{L}_{N^{\pm}}\Phi^{\pm}(x) \\ \{\Phi^{\mp}(x),\ H_{\pm}[N^{\pm}]\ \} &= 0, \end{aligned} \tag{12}$$

Thus, on the set of variables $\Phi^{\pm}$, infintesmal gauge transformations act as diffeomorphisms on S^1 and there is a split of the constraints and the phase space variables into commuting '+' and '-' parts which correspond to the usual right and left moving sectors of free scalar field theory. The action of the constraints on the embedding variables $X^{\pm}(x)$ preserves this split:

$$\{X^{\pm}(x), H_{\pm}[N^{\pm}]\} = N^{\pm}(X^{\pm})', \tag{13}$$

$$\{X^{\mp}(x), H_{\pm}[N^{\pm}]\} = 0. \tag{14}$$

Indeed, the above equations seem to indicate that infinitesimal gauge transformations, once again, act as diffeomorphisms on S^1; however, as we shall see in the next subsection, this interpretation is not strictly true for equations (13), (14) due to the non- existence of global, single valued coordinates on S^1.

2.3 Consequences of spatial topology $= S^1$.

2.3.1 Conditions on the canonical variables.

S^1 does not admit a global single valued coordinate system. However, at the cost of introducing appropriate periodic/quasiperiodic boundary conditions on the fields we may choose x to be the standard angular coordinate, $x \in [0, 2\pi]$ with the identification $x = 0 \sim x = 2\pi$. The Minkowskian coordinates $X^A = (T, X)$ in the action (1) are chosen so that $T \in (-\infty, \infty)$,$X \in (-\infty, \infty)$ with the identifications $X \sim X + 2\pi$.

The above specifications on x, X imply the following conditions on the canonical embedding variables and the Lagrange multipliers:

(i) $X^{\pm}(2\pi) - X^{\pm}(0) = 2\pi$.

(ii) Any two sets of embedding data $(X_1^+(x), X_1^-(x))$ and $(X_2^+(x), X_2^-(x))$ are to be identified if there exists an integer m such that $X_1^+(x) = X_2^+(x) + 2m\pi\ \forall\ x \in\ [0, 2\pi]$ and $X_1^-(x) = X_2^-(x) - 2m\pi\ \forall\ x \in\ [0, 2\pi]$.

(iii) $\Pi_{\pm}(x), N^{\pm}(x)$ and their spatial derivatives to all orders, as well as the spatial derivatives to all orders of the embedding coordinates $X^{\pm}(x)$ are periodic on $[0, 2\pi]$ with period 2π. This follows from the 1+1 Hamiltonian decomposition of (3) and the fact that $\frac{\partial X^A}{\partial x^\alpha}$ in equation (4) is single valued on $S^1 \times R$.

An additional "non-degeneracy" condition arises from (4):
(iv) $\pm(X^{\pm})' > 0$.

Since f in (1) is a single valued function on $S^1 \times R$, it follows that the matter phase space variables, (f, π_f) and their spatial derivatives to all orders are also periodic functions on $[0, 2\pi]$. Note also that the delta function $\delta(x, y)$ in (8), (9) is periodic in both its arguments.

2.3.2 Finite gauge transformations.

Whereas equation (12) implies that finite gauge transformations act on $(\Pi_{\pm}, Y^{\pm})$ as spatial diffeomorphisms on S^1, as remarked earlier the case of the embedding variables $X^{\pm}$ is more subtle as $X^{\pm}$ are not single valued fields on S^1 by virtue of **(i)**, section 2.3.1. Therefore, evolution of $X^{\pm}$ under the flow generated by the constraints is better understood in terms of transformations on the universal cover of S^1 as follows.

Unwind S^1 to its universal cover $\mathbf{R}$. Quasi-periodic boundary conditions obeyed by the embeddings suggest that their extension to $\mathbf{R}$ satisfies:

$$X^{\pm}_{ext}(x \pm 2n\pi) \; := \; X^{\pm}(x) \; \pm \; 2n\pi \tag{15}$$

where $x \in [0, 2\pi]$ and $n \in \mathbf{Z}$. The vector fields $N^{\pm}(x)$ on S^1 extend to periodic vector fields $N^{\pm}_{ext}$ on $\mathbf{R}$ so that $N^{\pm}_{ext}(x+2n\pi) \; = \; N^{\pm}(x), \; x \in [0, 2\pi]$. Let the 1 parameter family of (periodic) diffeomorphisms of $\mathbf{R}$ generated by $N^{\pm}_{ext}$ be denoted by $\phi(N^{\pm}_{ext}, t)$. Then it is straightforward to check that the finite transformations generated by the constraints on $X^{\pm}(x)$ are labelled by $\phi[N^{\pm}_{ext}, t]$ and act as follows:

$$\begin{aligned} (\alpha_{\phi[N^{\pm}_{ext},t]} X^{\pm})(x) \; &= \; X^{\pm}_{ext}(\phi[N^{\pm}_{ext}, t](x)) \; \forall \; x \in [0, 2\pi] \\ (\alpha_{\phi[N^{\pm}_{ext},t]} X^{\mp})(x) \; &= \; X^{\mp}(x) \; \forall \; x \in [0, 2\pi] \end{aligned} \tag{16}$$

Here $\alpha_{\phi[N^{\pm}_{ext},t]}$ is the flow generated by Hamiltonian vector field of $H_{\pm}[N^{\pm}]$.

It is also straightforward to see that the action of finite gauge transformations on the phase space variables $\Phi^{\pm} \in \{Y^{\pm}, \Pi_{\pm}\}$ can equally well be written in terms of the action of the periodic diffeomorphisms $\phi[N^{\pm}_{ext}, t]$ on the periodic extensions $\Phi^{\pm}_{ext}$ as

$$\begin{aligned} (\alpha_{\phi[N^{\pm}_{ext},t]} \Phi^{\pm})(x) \; &= \; \Phi^{\pm}_{ext}(\phi[N^{\pm}_{ext}, t](x)) \; \forall \; x \in [0, 2\pi] \\ (\alpha_{\phi[N^{\pm}_{ext},t]} \Phi^{\mp})(x) \; &= \; \Phi^{\mp}(x) \; \forall \; x \in [0, 2\pi] \end{aligned} \tag{17}$$

Here $\Phi^{\pm}_{ext}(x + 2n\pi) \; = \; \Phi^{\pm}(x) \; \forall x \in [0, 2\pi], n \in \mathbf{Z}$.

Since $\phi[N^{\pm}_{ext}, t]$, $\forall (N^{\pm}_{ext}, t)$ range over all periodic diffeomorphisms of $\mathbf{R}$ connected to identity, we label every finite gauge transformation by a pair of such diffeomorphisms (ϕ^{+}, ϕ^{-}) so that the Hamiltonian flows generated by $H_{\pm}$ are denoted by $\alpha_{\phi^{\pm}}$. To summarise: Let $\Psi^{\pm}(x) \in (X^{\pm}(x), \Pi_{\pm}(x), Y^{\pm}(x))$

and let its appropriate quasiperiodic/periodic extension on $\mathbf{R}$ be $\Psi^{\pm}_{ext}$. Then we have that, $\forall x \in [0, 2\pi]$,

$$\begin{aligned} (\alpha_{\phi^{\pm}}\Psi^{\pm})(x) &= \Psi^{\pm}_{ext}(\phi^{\pm}(x)) \\ (\alpha_{\phi^{\pm}}\Psi^{\mp})(x) &= \Psi^{\mp}(x). \end{aligned} \tag{18}$$

Equations (18) imply a left representation of the group of periodic diffeomorphisms of $\mathbf{R}$ by the Hamiltonian flows corresponding to finite gauge transformations:

$$\alpha_{\phi_1^{\pm}}\alpha_{\phi_2^{\pm}} = \alpha_{\phi^{\pm}\circ\phi_2^{\pm}} \tag{19}$$

$$\alpha_{\phi_1^{\pm}}\alpha_{\phi_2^{\mp}} = \alpha_{\phi_2^{\mp}}\alpha_{\phi_1^{\pm}}. \tag{20}$$

We emphasize that the extended fields are only formal constructs which are useful for interpreting gauge transformations in terms periodic diffeomorphisms of $\mathbf{R}$. The spatial slice is always S^1 coordinatized by $x \in [0, 2\pi]$ with boundary points identified.

2.4 Dirac Observables

Since finite gauge transformations act as periodic diffeomorphisms of $\mathbf{R}$, it follows, directly, that the integral over $x \in [0, 2\pi]$ of any periodic scalar density constructed solely from the phase space variables, is an observable.

An analysis of the Hamiltonian equations [24] shows that the relation between solutions $f(X^+, X^-)$ of the flat spacetime wave equation and canonical data $(Y^{\pm}, X^{\pm})$ on the constraint surface is

$$\pm 2\frac{\partial f}{\partial X^{\pm}} = \frac{Y^{\pm}}{(X^{\pm})'}. \tag{21}$$

Here f is evaluated at the spacetime point (X^+, X^-) defined by the canonical data. Recall that any solution $f(X^+, X^-)$ to the free scalar field equation is of the form

$$f(X^+, X^-) = \mathbf{q} + \mathbf{p}\frac{(X^+ + X^-)}{2} - i\sum_{n=1}^{\infty}(\mathbf{a}_{(+)\mathbf{n}}e^{-inX^+} + \mathbf{a}_{(-)\mathbf{n}}e^{-inX^+} + \text{c.c}), \tag{22}$$

where c.c. stands for 'complex conjugate'. Equations (21) and (22) yield an interpretation for the Dirac observables constructed below.

2.4.1 Mode functions.

From (21) and (22) and the remarks above, it follows that

$$a_{(\pm)n} = \int_{S^1} dx Y^{\pm}(x) e^{inX^{\pm}(x)},\ n \in \mathbf{Z},\ n > 0 \tag{23}$$

(and their complex conjugates, $a^*_{(\pm)n}$,) are Dirac observables which correspond to the mode functions $\mathbf{a}_{(\pm)\mathbf{n}}$ of equation (22). These observables form the (Poisson) algebra,

$$\begin{aligned}\{a_n, a_m*\} &= -4\pi i n \delta_{n,m}, \\ \{a_n, a_m\} &= 0, \\ \{a_n*, a_m*\} &= 0.\end{aligned} \tag{24}$$

2.4.2 Zero modes.

The quantities $\mathbf{q}, \mathbf{p}$ in equation (22) are referred to as zero modes of the scalar field and are also realizable as Dirac observables which are canonically conjugate to each other [24]. Indeed, it is straightforward to see from (21), (22) that $\mathbf{p}$ corresponds to $p := \int_{S^1} dx Y^+(x) = \int_{S^1} dx Y^-(x)$. However, the degree of freedom corresponding to $\mathbf{q}$ is absent in the phase space coordinates $(X^\pm, \Pi_\pm, Y^\pm)$ as a result of $Y^\pm$ only containing derivatives of f (see equation (21)).

Our aim in this work is to construct a triangulation independent polymer quantization of a generally covariant field theoretic model. Issues related to the construction of zero modes (which are anyway mechanical (as opposed to field theoretic) degrees of freedom) as Dirac observables serve to distract from this aim. Hence we shall switch off the zero modes by setting $\mathbf{q} = \mathbf{p} = 0$. Since $\mathbf{q}$ and $\mathbf{p}$ are canonically conjugate, this can be done consistently. In the free scalar field action (1) this corresponds to limiting the space of all scalar fields by the conditions $\mathbf{q} = \int_{S^1} dX f(T,X) = 0$ and $\mathbf{p} = \int_{S^1} dX \frac{\partial f(T,X)}{\partial T} = 0$. In the canonical description of PFT in terms of $(\Pi_\pm, X^\pm, Y^\pm)$, since $\mathbf{q}$ does not appear, we only need to set the quantity

$$p := \int_{S^1} dx Y^+(x) = \int_{S^1} dx Y^-(x) = 0. \tag{25}$$

Since, as can easily be checked, p commutes with $(\Pi_\pm, X^\pm, Y^\pm)$ as well as the constraints (10), it is consistent to impose (25).

To summarize: The system we consider in this work is PFT on $S^1 \times R$ with the zero modes switched off. The phase space variables are $(\Pi_\pm, X^\pm, Y^\pm)$ subject to the conditions of section 2.3.1. The symplectic structure is given by (8) and (9) and the constraints by (10). The degrees of freedom of the theory reside entirely in the mode coefficients $\mathbf{a}_{(\pm)\mathbf{n}}, \mathbf{a}_{(\pm)\mathbf{n}}{}^*$ (22) which are expressed as the functions $a_{(\pm)n}, a^*_{(\pm)n}$ on phase space via (23).

2.4.3 Conformal Isometries.

Free scalar field theory in 1+1 dimensions (1) is conformally invariant. As a consequence the generators of conformal isometries in PFT are also Dirac observables (for details, see Reference [24]). Consider the conformal isometry

generated by the conformal Killing field $\vec{U}$ on the Minkowskian cylinder. Let $\vec{U}$ have the components $(U^+(X^+), U^-(X^-))$ in the (X^+, X^-) coordinate system. $U^\pm$ are periodic functions of $X^\pm$ by virtue of the fact that $\vec{U}$ is smooth vector field on the flat spacetime $S^1 \times R$ These components of $\vec{U}$ naturally correspond to the functions $(U^+(X^+(x)), U^-(X^-(x)))$ on the phase space of PFT. The Dirac observable corresponding to the generator of conformal transformations associated with $\vec{U}$ is given by

$$\Pi_\pm[U^\pm] \ = \ \int_{S^1} \Pi_\pm(x) U^\pm(X^\pm(x)) \tag{26}$$

These observables generate a Poisson algebra isomorphic to that of the commutator algebra of conformal Killing fields:

$$\begin{aligned} \{\Pi_\pm[U^\pm], \Pi_\pm[V^\pm]\} \ &= \ \Pi[[U,V]^\pm] \\ \{\Pi_\pm[U^\pm], \Pi_\mp[V^\mp]\} \ &= 0. \end{aligned} \tag{27}$$

Here $[U,V]^\pm$ refer to the $\pm$ components of the commutator of the spacetime vector fields $\vec{U}, \vec{V}$. $[U,V]^\pm$ define functions of the embedding variables $X^\pm(x)$ in the manner described above.

Note that these observables are weakly equivalent, via the constraints (10) to quadratic combinations of the mode functions [24]. In the standard Fock representation of quantum theory (see for e.g. Reference [1]), these quadratic combinations are nothing but the generators of the Virasoro algebra.

As we shall see, the polymer quantization of PFT provides a representation for the finite canonical transformations generated by $\Pi^\pm[U^\pm]$. For future reference, it is straightforward to check that the Hamiltonian flow, $\alpha_{(\Pi_\pm[U^\pm],t)}$ generated by $\Pi_\pm[U^\pm]$ leaves the matter sector of phase space untouched and acts on the embedding variables $X^\pm$ as

$$\alpha_{(\Pi_\pm[U^\pm],t)} X^\pm(x) = (\phi_{(\vec{U},t)} X^\pm)(x). \tag{28}$$

Here $\phi_{(\vec{U},t)}$ denotes the one parameter family of conformal isometries generated by the conformal Killing field $\vec{U}$ on spacetime. $\phi_{(\vec{U},t)}$ maps the spacetime point (X^+, X^-) to $\phi_{(\vec{U},t)} X^\pm$ and hence maps the spatial slice defined by the canonical data $X^\pm(x)$ is mapped to the new slice (and hence the new canonical data) $(\phi_{(\vec{U},t)} X^\pm)(x)$.

$\phi_{(\vec{U},t)}$ ranges over all conformal isometries connected to identity. Any such conformal isometry ϕ_c is specified by a pair of functions $\phi_c^\pm$ so that $\phi_c(X^+, X^-) := (\phi_c^+(X^+), \phi_c^-(X^-))$. Invertibilty of ϕ_c together with connectedness with identity implies that

$$\frac{d\phi_c^\pm}{dX^\pm} > 0, \tag{29}$$

and the cylindrical topology of spacetime implies that

$$\phi_c^{\pm}(X^{\pm} \pm 2\pi) = \phi_c^{\pm}(X^{\pm}) \pm 2\pi. \tag{30}$$

Thus, we may denote the Hamiltonian flows which generate conformal isometries by α_{ϕ_c} or, without loss of generality, by $\alpha_{\phi_c^{\pm}}$ with $\alpha_{\phi_c^{\pm}}$ acting trivially on the $\mp$ sector.

To summarise:$\alpha_{\phi_c^{\pm}}$ leave the matter variables untouched, so that

$$\alpha_{\phi_c^{\pm}} Y^{\pm}(x) = Y^{\pm}(x), \quad \alpha_{\phi_c^{\pm}} Y^{\mp}(x) = Y^{\mp}(x), \tag{31}$$

and act on $X^{\pm}(x)$ as

$$\alpha_{\phi_c^{\pm}} X^{\pm}(x) = \phi_c^{\pm}(X^{\pm}(x)), \quad \alpha_{\phi_c^{\pm}} X^{\mp}(x) = X^{\mp}(x). \tag{32}$$

Further, since $\Pi_{\pm}[U^{\pm}]$ are observables which commute strongly with the constraints, the corresponding Hamiltonian flows are gauge invariant. This translates to the condition that for all

$$\begin{aligned} \alpha_{\phi_c^{\pm}} \circ \alpha_{\phi^+} &= \alpha_{\phi^+} \circ \alpha_{\phi_c^{\pm}} \\ \alpha_{\phi_c^{\pm}} \circ \alpha_{\phi^-} &= \alpha_{\phi^-} \circ \alpha_{\phi_c^{\pm}} \end{aligned} \tag{33}$$

where as before $\phi^{\pm}$ label finite gauge transformations.

3 Polymer Quantum Kinematics.

3.1 Preliminaries.

As in LQG, the polymer quantization is based on suitably defined "holonomies" and the polymer Hilbert space is spanned by suitably defined "charge network" states. In view of the correspondence between finite gauge transformations and periodic diffeomorphisms of $\mathbf{R}$, it is useful to to define periodic and quasiperiodic extensions of charge network labels. Hence we define the following.

Definition 1 : A charge-network s is specified by the labels $(\gamma(s), (j_{e_1}, ..., j_{e_n}))$ consisting of a graph $\gamma(s)$ (by which we mean a finite collection of closed, non-overlapping(except in boundary points) intervals which cover $[0, 2\pi]$) and 'charges' $j_e \in \mathbf{R}$ assigned to each interval e. (Note that $j_e = 0$ is allowed.) Equivalence classes of charge- networks are defined as follows. The charge- network s' is said to be finer than s iff (a) every edge of $\gamma(s)$ is identical to, or composed of, edges in $\gamma(s')$ (b) the charge labels of identical edges in $\gamma(s), \gamma(s')$ are identical and the charge labels of the edges of $\gamma(s')$ which compose to yield an edge of $\gamma(s)$ are identical and equal to that of their union in $\gamma(s)$. Two charge- networks are equivalent if there exists a

charge- network finer than both. Hence we can represent each equivalence class by a unique representative s such that no two adjacent edges have the same charge. However, unless otherwise mentioned, s will not necessarily denote this unique choice.

Definition 2: The periodic extension of the charge- network s to $\mathbf{R}$ is denoted by s_{ext} and defined as follows.

Given a graph γ as in Definition 1 above, $T_N(\gamma)$ denotes the translation of γ by $2N\pi$, i.e. $T_N(\gamma)$ lies in $[2N\pi, 2(N+1)\pi]$. We define the *extension* of γ to $\mathbf{R}$ as $\gamma_{ext} = \cup_{N\in\mathbf{Z}} T_N(\gamma)$. The *restriction* of γ_{ext} to any interval $I \subset \mathbf{R}$ is denoted by $\gamma_{ext}|_I$ so that $\gamma_{ext}|_{[0,2\pi]} = \gamma$.

Given a charge network $s = (\gamma(s), (j_{e_1}, ..., j_{e_n}))$, s_{ext} is specified by the graph $\gamma(s_{ext}) := \gamma(s)_{ext}$ ($\gamma(s)_{ext}$ denotes the extension of $\gamma(s)$ to $\mathbf{R}$) and charge labels for each edge of $\gamma(s_{ext})$ which are such that $T_N(\gamma(s)) \subset \gamma(s_{ext})$ has the same set of charges which are on γ. Thus

1. On any closed interval $I_N = [2N\pi,\ 2(N+1)\pi]$, $N \in \mathbf{Z}$, $\gamma(s_{ext})|_{I_N} \cong \gamma(s)$.
2. The set of charges on $\gamma(s_{ext})|_{I_N}$ is $(j_{e_1}, ..., j_{e_n})$.

We refer to $s_{ext}|_{[0,2\pi]}$ as the restriction of s_{ext} to $[0, 2\pi]$ so that $s_{ext}|_{[0,2\pi]} = s$.

Definition 3: The quasi- periodic extension of the charge- network s to $\mathbf{R}$ is denoted by $\bar{s}_{ext}$ and defined as follows. Given a charge network $s = (\gamma(s), (j_{e_1}, ..., j_{e_n}))$, $\bar{s}_{ext}$ is specified by the graph $\gamma(\bar{s}_{ext}) := \gamma(s)_{ext}$ and charge labels for each edge of $\gamma(\bar{s}_{ext})$ which are such that $T_N(\gamma(s)) \subset \gamma(\bar{s}_{ext})$ has the set of charges which are on γ augmented by $2N\pi$. Thus

1. On any closed interval $I_N = [2N\pi,\ 2(N+1)\pi]$, $N \in \mathbf{Z}$, $\gamma(\bar{s}_{ext})|_{I_N} \cong \gamma(s)$.
2. The set of charges on $\gamma(\bar{s}_{ext})|_{I_N}$ is $(j_{e_1} + 2N\pi, ..., j_{e_n} + 2N\pi)$.

Definition 4: The action of periodic diffeomorphisms on $\gamma_{ext}, s_{ext}, \bar{s}_{ext}$ may be defined as follows. Any periodic diffeomorphism ϕ of $\mathbf{R}$ commutes with the 2π translations, T_N and hence has a natural action on the extension γ_{ext} of the graph γ. Denote the resulting graph by $\phi(\gamma_{ext})$ and let the edge $\phi(e) \in \phi(\gamma_{ext})$ be the image, by ϕ of the edge $e \in \gamma_{ext}$. The action of ϕ on the extensions $s_{ext}, \bar{s}_{ext}$ is defined by
(i) mapping the underlying graph $\gamma(s)_{ext}$ to $\phi(\gamma(s)_{ext})$
(ii) labelling the edge $\phi(e) \in \phi(\gamma(s)_{ext})$ by the same charge as the edge $e \in \gamma(s)_{ext}$ so that $k_{\phi(e)} = k_e$.

Denote the resulting periodic/quasiperiodic charge networks on $\mathbf{R}$ by $\phi(s_{ext})/\phi(\bar{s}_{ext})$

3.2 Embedding sector.

3.2.1 The *- Algebra

The elementary variables which generate the *-Poisson algebra are, $X^+(x), T_{s^+}[\Pi_+]$, $X^-(x), T_{s^-}[\Pi_-]$. Here $T_{s^\pm}[\Pi_\pm]$ are the holonomy- type functions associated with the charge networks $s^\pm$ given by

$$T_{s^\pm}[\Pi_\pm] \;=\; \prod_{e^\pm\in\gamma(s^\pm)} \exp[-ik^\pm_{e^\pm}\int_{e^\pm}\Pi_\pm]. \tag{34}$$

The only non- trivial Poisson brackets are:

$$\begin{aligned}
\{X^\pm(x), T_{s^\pm}[\Pi_\pm]\} &= -ik^\pm_{e^\pm} T_{s^\pm}[\Pi_\pm] \; if \; x \in \text{Interior}(e^\pm) \\
&= -\tfrac{i}{2}(k^\pm_{e^\pm_{I^\pm}} + k^\pm_{e^\pm_{(I+1)^\pm}}) T^E_{s^\pm}[\Pi_\pm] \; if \; x \in e^\pm_{I^\pm} \cap e^\pm_{(I+1)^\pm} \;\; 1 \leq I^\pm \leq (n-1)^\pm \\
\{X^\pm(0), T_{s^\pm}[\Pi_\pm]\} &= \{X^\pm(2\pi), T_{s^\pm}[\Pi_\pm]\} \;=\; -\tfrac{i}{2}(k^\pm_{e^\pm_1} + k^\pm_{e^\pm_{n^\pm}}) T_{s^\pm}[\Pi_\pm],
\end{aligned} \tag{35}$$

where the last Poisson bracket uses the periodicity of delta function. The *-relations are given by

$$\begin{aligned}
(X^\pm(x))^* &= X^\pm(x) \; \forall \; x \in \; [0, 2\pi] \\
T_{s^\pm}[\Pi_\pm]^* &= T_{-s^\pm}[\Pi_\pm], \; -s^\pm = (\gamma(s^\pm), (-k^\pm_{e^\pm_1}, ..., -k^\pm_{e^\pm_{n^\pm}}))
\end{aligned} \tag{36}$$

The action of finite gauge transformations on these elementary functions is as follows (we only analyse the right-moving sector; the analysis of the left moving sector is identical).

From equation (18) we have,

$$\alpha_{\phi^+} T_{s^+}[\Pi_+] \;=\; T_{s^+}[(\phi^+)_* \Pi_+]. \tag{37}$$

It is straightforward to check, using the periodicity of ϕ^+, Π_+, s^+_{ext} and the various definitions in section 3.1 that

$$T_{s^+}[(\phi^+)_*\Pi_+] = T_{\phi^+(s^+_{ext})|_{[0,2\pi]}}[\Pi_+]. \tag{38}$$

Finite gauge transformations act on $X^\pm$ as in equations (16), (18). To summarise, under finite gauge transformations the generators of the Poisson algebra transform as:

$$\begin{aligned}
\alpha_{\phi^\pm}(X^\pm)(x) &= X^\pm_{ext}((\phi^\pm)(x)) \;=\; X^\pm(y) \; \pm 2\pi \; N \text{ if } (\phi^\pm)(x) = y + 2\pi N \; y \in \; [0, 2\pi] \\
\alpha_{\phi^\mp}(X^\pm)(x) &= X^\pm(x) \\
\alpha_{\phi^\pm}(T_{s^\pm}[\Pi^\pm]) &= T^\pm_{\phi^\pm(s^\pm_{ext})|_{[0,2\pi]}}[\Pi^\pm] \\
\alpha_{\phi^\mp}(T_{s^\pm}[\Pi^\pm]) &= T_{s^\pm}[\Pi^\pm]
\end{aligned} \tag{39}$$

3.2.2 Representation of the *- Algebra

Denote the kinematic Hilbert space for the $\pm$ embedding sectors by $\mathcal{H}_E^{\pm}$. $\mathcal{H}_E^{\pm}$ is the closure of the span of the orthonormal basis of embedding 'charge network states'. Each such state is labelled by a charge network $s^{\pm}$ and denoted by $T_{s^{\pm}}$. [2] The inner product is

$$< T_{s^{\pm}},\ T_{s'^{\pm}} > =\ \delta_{s^{\pm},s'^{\pm}} \tag{40}$$

where $\delta_{s^{\pm},s'^{\pm}}$ is a Kronecker delta function which is unity when the two charge networks are identical and vanishes otherwise.

The '$\pm$' sector operators corresponding to the elementary functions of the previous section are denoted by $\hat{X}^{\pm}(x), \hat{T}_{s^{\pm}}$. $\hat{T}_{s^{\pm}}$ acts on the charge network states as:

$$\hat{T}_{s^{\pm}} T_{s'^{\pm}}\ :=\ T_{s^{\pm}+s'^{\pm}} \tag{41}$$

where $s^{\pm}+s'^{\pm}$ is the charge network obtained by dividing $\gamma(s^{\pm})$, $\gamma(s'^{\pm})$ into maximal, non-overlapping (upto boundary points) intervals and assigning charge $k^{\pm}_{e^{\pm}}\ +\ k^{\pm}_{e'^{\pm}}$ to $e^{\pm}\cap e'^{\pm}$ where $e^{\pm}\in\gamma(s^{\pm})$, $e'^{\pm}\in\gamma(s^{\pm}_1)$.

The action of $\hat{X}^{\pm}(x)$ is:

$$\hat{X}^{\pm}(x)T_{s^{\pm}}\ :=\ \lambda_{x,s^{\pm}}T_{s^{\pm}}, \tag{42}$$

where, for $\gamma(s^{\pm})$ with $n^{\pm}$ edges,

$$\begin{aligned}
\lambda_{x,s^{\pm}} &:=\ \hbar k^{\pm}_{e^{\pm}_{I^{\pm}}}\ T_{s^{\pm}}\ if\ x\in \text{Interior}(e^{\pm}_{I^{\pm}})\ 1\leq I^{\pm}\leq n^{\pm} \\
&:=\ \frac{\hbar}{2}(k^{\pm}_{e^{\pm}_{I^{\pm}}}\ +\ k^{\pm}_{e^{\pm}_{(I+1)^{\pm}}})T_{s^{\pm}}\ if\ x\in e^{\pm}_{I^{\pm}}\cap e^{\pm}_{(I+1)^{\pm}}\ 1\leq I^{\pm}\leq (n-1)^{\pm}
\end{aligned} \tag{43}$$

$$\begin{aligned}
&:=\ \frac{\hbar}{2}(k^{\pm}_{e^{\pm}_{n^{\pm}}}\ \mp\ \frac{2\pi}{\hbar}\ +\ k^{\pm}_{e^{\pm}_1})T_{s^{\pm}}\ if\ x=0 \\
&:=\ \frac{\hbar}{2}(k^{\pm}_{e^{\pm}_1}\ \pm\ \frac{2\pi}{\hbar}\ +\ k^{\pm}_{e^{\pm}_{n^{\pm}}})T_{s^{\pm}}\ if\ x=2\pi
\end{aligned} \tag{44}$$

The last two equations, (44), implement the boundary condition $X^{\pm}(2\pi)-X^{\pm}(0)=\pm 2\pi$ (see **(i)** of section 2.3.1.

It is straightforward to check that equations (41),(42),(43),(44) provide a representation of the Poisson bracket algebra (35) so that quantum commutators equal $i\hbar$ times the Poisson brackets. It is also straightforward to verify that the *- relations (36) on $\hat{X}^{\pm}(x), \hat{T}_{s^{\pm}}$ are implemented by the inner product (40) so that $\hat{X}^{\pm}(x)$ are self adjoint and $\hat{T}_{s^{\pm}}$ are unitary.

[2] More precisely, the labelling is by the equivalence class of $s^{\pm}$ as in Definition 1, section 3.1

3.2.3 Unitary representation of finite gauge transformations.

Since the Hamiltonian flows of $\alpha_{\phi^{\pm}}$ (18) are real, the corresponding quantum operators $\hat{U}(\phi^{\pm})$ must be unitary. Equations (18), (19) imply that this unitary representation must satisfy

$$\begin{aligned}
\hat{U}^{\pm}(\phi_1^{\pm})\hat{U}^{\pm}(\phi_2^{\pm}) &= \hat{U}^{\pm}(\phi_1^{\pm}\circ\phi_2^{\pm}) \\
\hat{U}^{\pm}(\phi^{\pm})\hat{X}^{\pm}(x)\hat{U}^{\pm}(\phi^{\pm})^{-1} &= \hat{X}^{\pm}(y^{\pm}) \pm 2\pi N^{\pm} \\
\hat{U}^{\pm}(\phi^{\pm})\hat{T}_{s^{\pm}}\hat{U}^{\pm}(\phi^{\pm})^{-1} &= \hat{T}_{\phi^{\pm}(s^{\pm})_{ext}|[0,2\pi]}.
\end{aligned} \tag{45}$$

where $\phi^{\pm}(x) = y^{\pm} + 2\pi N^{\pm}$, with $y^{\pm}\in[0,2\pi]$ and $N^{\pm}\in\mathbf{Z}$.
We define the action of $\hat{U}(\phi^{\pm})$ to be

$$\begin{aligned}
\hat{U}^{\pm}(\phi^{\pm})T_{s^{\pm}} &:= T_{\phi(\overline{s}^{\pm}_{ext})|_{[0,2\pi]}} \\
\hat{U}^{\mp}(\phi^{\mp})T_{s^{\pm}} &:= T_{s^{\pm}}.
\end{aligned} \tag{46}$$

The appearance of the quasi-periodic extensions $\bar{s}^{\pm}_{ext}$ of the charge networks $s^{\pm}$ (see Definition 3, section 3.1) in the first equation above may be anticipated from the quasi- periodic nature of the embedding variables $X^{\pm}(x)$ (15). Unitarity of $\hat{U}^{\pm}(\phi^{\pm})$ follows straightforwardly:

$$\begin{aligned}
<\ \hat{U}^{\pm}(\phi^{\pm})T_{s_1^{\pm}}\ ,\ \hat{U}^{\pm}(\phi^{\pm})T_{s_2^{\pm}}\ > &= <\ T_{\phi(\overline{s}_1^{ext\pm}|_{[0,2\pi]})}\ ,\ T_{\phi(\overline{s}_2^{ext\pm}|_{[0,2\pi]})}\ > \\
&= \delta_{\phi^{\pm}(\overline{s}_1^{ext\pm}|_{[0,2\pi]}),\ \phi^{\pm}(\overline{s}_2^{ext\pm}|_{[0,2\pi]})}\ \forall\ \phi^{\pm} \\
&= \delta_{s_1^{\pm},s_2^{\pm}}
\end{aligned} \tag{47}$$

where we have used the fact that two charge-networks are equal on $[0,2\pi]$ iff their extensions are equal.

From equation (46) and Definitions 3,4 of section 3.1, it follows that

$$\begin{aligned}
\hat{U}^{\pm}(\phi_1^{\pm})\hat{U}^{\pm}(\phi_2^{\pm})\ T_{s^{\pm}} &= T_{\phi_1^{\pm}(\overline{\phi_2^{\pm}(\overline{s^{\pm}}_{ext})|_{[0,2\pi]}})_{ext}|_{[0,2\pi]}} \\
&= T_{\phi_1^{\pm}(\phi_2^{\pm}(\overline{s}^{\pm}_{ext}))|_{[0,2\pi]}} \\
&= T_{(\phi_1^{\pm}\circ\phi_2^{\pm})(\overline{s}^{\pm}_{ext})|_{[0,2\pi]}} \\
&= \hat{U}^{\pm}(\phi_1^{\pm}\circ\phi_2^{\pm})T_{s^{\pm}},
\end{aligned} \tag{48}$$

thus verifying the first relation in (45).

Next, we turn to the second relation of (45). We sketch the proof for the '+' sector; the proof for the '-' sector is on similar lines. From (46) and (42) we have that:

$$\begin{aligned}
\hat{U}^{+}(\phi^{+})\hat{X}^{+}(x)\hat{U}^{+}(\phi^{+})^{-1}\ T_{s^{+}} &= \hat{U}^{+}(\phi^{+})\hat{X}^{+}(x)T_{(\phi^{+})^{-1}(\overline{s}^{+}_{ext}|_{[0,2\pi]}} \\
&= \lambda_{x,(\phi^{+})^{-1}(\overline{s}^{+}_{ext}|_{[0,2\pi]})}T_{s^{+}}.
\end{aligned} \tag{49}$$

It is straightforward to see that

$$\lambda_{x,(\phi^+)^{-1}(\overline{s}^+_{ext}|_{[0,2\pi]})} \;=\; \lambda_{y^+,s^+} + 2\pi N^+, \tag{50}$$

which via equation (42) obtains the desired result.

Finally, we turn to the last relation of (45). Once again, we sketch the proof for the '+' sector; the '-' sector proof follows analogously. We want to show that

$$\hat{U}^+(\phi^+)\hat{T}_{s^+}\hat{U}^+((\phi^+)^{-1}) \;=\; \hat{T}_{\phi^+(s^+_{ext})|_{[0,2\pi]}}. \tag{51}$$

Since charge network states form an orthonormal basis in the Hilbert space, it follows that (51) is equivalent to the condition that $\forall s_1^+, s_2^+$

$$\langle T_{(\phi^+)^{-1}(\overline{s^+}_1)_{ext}|_{[0,2\pi]}}|\; \hat{T}_{s^+}|\; T_{(\phi^+)^{-1}(\overline{s^+}_2)_{ext}|_{[0,2\pi]}}\rangle = \langle T_{s_1^+}|\; \hat{T}_{\phi^+(s^+_{ext})|_{[0,2\pi]}}|T_{s_2^+}\rangle, \tag{52}$$

which from equation (41) is, in turn, equivalent to the equation

$$\delta_{(\phi^+)^{-1}(\overline{s^+}_1)_{ext}|_{[0,2\pi]},s^+ + (\phi^+)^{-1}(\overline{s^+}_2)_{ext}|_{[0,2\pi]}} \;=\; \delta_{s_1^+,\phi^+(s^+_{ext})|_{[0,2\pi]}+s_2^+}. \tag{53}$$

However, (suppressing the '+' superscript), we have that

$$\begin{aligned}\delta_{\phi^{-1}(\overline{s}_1)_{ext}|_{[0,2\pi]},s+\phi^{-1}(\overline{s}_2)_{ext}|_{[0,2\pi]}} &= \delta_{\phi^{-1}(\overline{s}_1)_{ext},s_{ext}+\phi^{-1}(\overline{s}_2)_{ext}}\\ &= \delta_{(\overline{s}_1)_{ext},\phi(s_{ext})+(\overline{s}_2)_{ext}}\\ &= \delta_{(s_1)_{ext},\phi(s_{ext})+(s_2)_{ext}}\\ &= \delta_{s_1,\phi(s_{ext})|_{[0,2\pi]}+s_2},\end{aligned} \tag{54}$$

thus proving (51).

3.3 Matter sector.

3.3.1 The *- Algebra.

The *- Algebra is generated by the operators corresponding to the classical holonomies $W_{s^\pm}[Y^\pm]$ which are defined as

$$W_{s^\pm}[Y^\pm] \;=\; \exp[i \sum_{e^\pm\in E(\gamma(s^\pm))} l^\pm_{e^\pm}\int_{e^\pm} Y^\pm]. \tag{55}$$

Here $s^\pm \;:=\; \{\gamma(s^\pm),\; (l^\pm_{e^\pm_1},...,l^\pm_{e^\pm_{m^\pm}})\;\}$ are charge- networks. The algebra for the holonomy operators is the analog of the Weyl algebra for linear quantum fields. Similar to that case, we need to first evaluate the Poisson brackets, $\{\sum_{e^\pm} l^\pm_{e^\pm}\int_{e^\pm} Y^\pm,\; \sum_{e'^\pm} l^\pm_{e'^\pm}\int_{e'^\pm} Y^\pm\}$, between the exponents of pairs of classical holonomies and then use the Baker- Campbell- Hausdorff Lemma [25] to define the algebra on the holonomy operators in quantum theory.

Let κ_e be the characteristic function associated with a closed interval e and denote the beginning and final points of e by $b(e)$ and $f(e)$ so that

$$\begin{aligned} \kappa_e(x) &= 1 \; if \; x \in \text{Interior}(e) \\ &= \tfrac{1}{2} \; if \; x = b(e) \; or \; f(e) \end{aligned} \tag{56}$$

$$\begin{aligned} &= \tfrac{1}{2} \; if \; x = 0 \text{ and } f(e) = 2\pi \\ &= \tfrac{1}{2} \; if \; x = 2\pi \text{and } b(e) = 0. \end{aligned} \tag{57}$$

Here, equations (57) follow from the periodicity of the delta function. From equation (9) it follows that

$$\{\int_{e^\pm} Y^\pm, \int_{e'^\pm} Y^\pm\} \;=\; \pm\alpha(e^\pm, e'^\pm) \;:=\; \pm(\kappa_{e'^\pm}|\partial_{e^\pm} \;-\; \kappa_{e^\pm}|\partial_{e'^\pm}), \tag{58}$$

where

$$\kappa_e|\partial_{e'} := \kappa_e(f(e')) \;-\; \kappa_e(b(e')), \tag{59}$$

so that

$$\{\sum_{e^\pm} l^\pm_{e^\pm} \int_{e^\pm} Y^\pm, \sum_{e'^\pm} l^\pm_{e'^\pm} \int_{e'^\pm} Y^\pm\} \;=\; \pm \sum_{e^\pm, e'^\pm} l^\pm_{e^\pm} l^\pm_{e'^\pm} \alpha(e^\pm, e'^\pm). \tag{60}$$

It follows that the 'Weyl algebra' of holonomy operators is:

$$\begin{aligned} \hat{W}(s^\pm)\hat{W}(s'^\pm) &= \exp[\mp\tfrac{i\hbar}{2}\alpha(s^\pm, s'^\pm)]\hat{W}(s^\pm + s'^\pm), \\ \hat{W}(s^\pm)^* &= \hat{W}(-s^\pm), \end{aligned} \tag{61}$$

where

$$\alpha(s^\pm, s'^\pm) \;:=\; \sum_{e^\pm \in \gamma(s^\pm)} \sum_{e'^\pm \in \gamma(s'^\pm)} l^\pm_e l^\pm_{e'^\pm} \alpha(e^\pm, e'^\pm), \tag{62}$$

with $\alpha(e, e')$ defined through equations (59) and (58). From the second equation of (9), it follows that the '+' and '-' holonomy operators commute, so that, once again, these sectors can be treated independently.

3.3.2 Representation of the *- Algebra.

It is convenient to define the quantum theory through the Gelfand- Naimark - Segal (GNS) construction [26]. The explicit operator action on the basis of charge network states is provided after we present the GNS state.

We define the GNS states $\omega_\pm$ on the $\pm$ holonomy algebras by specifying their action on the holonomy operators as follows:

$$\omega^\pm_M(\hat{W}(s^\pm)) \;=\; \delta_{s^\pm, \circ}. \tag{63}$$

Here '$\circ$' is the trivial charge network which may be represented by graph $\gamma(\circ)$ consisting of the single edge $e = [0, 2\pi]$ with vanishing charge $l^\pm_e = 0$.

The Kronecker delta function $\delta_{s^\pm,\circ}$ is unity iff $s^\pm = \circ$ and vanishes otherwise. It follows from the GNS construction that the corresponding GNS Hilbert spaces $\mathcal{H}^\pm_M$ are spanned by charge network states denoted by $W_{s^\pm}$. The inner product is

$$< W(s^\pm), W(s'^\pm) >_\pm \; = \; \delta_{s^\pm, s'^\pm} \tag{64}$$

and the action of the holonomy operators is

$$\hat{W}(s^\pm)W(s'^\pm) \; = \; \exp[\mp\frac{i\hbar\alpha(s^\pm, s'^\pm)}{2}]W(s^\pm + s'^\pm). \tag{65}$$

Here, as for the embedding sector, $s^\pm + s'^\pm$ is obtained by sub-dividing $s^\pm$ and $s'^\pm$ into maximal non-overlapping (upto boundary points) intervals and putting charges $l^\pm_e \; + \; l^\pm_{e'^\pm}$ on $e^\pm \cap e'^\pm$. ($e^\pm \in \; s^\pm$, $e'^\pm \in \; s'^\pm$). [3]

It is straightforward to check, explicitly, that equation (65) provides a representation for the first equation of (61). Verification of the second equation of (61) is equivalent to showing that $\forall s^\pm, s'^\pm, s''^\pm$,

$$< W(s'^\pm), \; (\hat{W}(s^\pm))^\dagger W(s''^\pm) \; >_\pm = < W(s'^\pm), \; \hat{W}(-s^\pm)W(s''^\pm) >_\pm . \tag{66}$$

Equation (66) follows straightforwardly from (64),(65). One needs to use the identity $\delta_{s^\pm, -s'^\pm + s''^\pm} = \delta_{s^\pm + s'^\pm, s''^\pm}$ and the easily verifiable fact that $\alpha(s^\pm, s'^\pm)$ is bilinear and antisymmetric in its arguments.

3.3.3 Unitary representation of finite gauge transformations.

Since $Y^\pm$ are periodic scalar densities, under finite gauge transformations their holonomies transform in a similar manner to those of the embedding momenta. Specifically, equation (18) in conjunction with the periodicity of $\phi^\pm, Y^\pm, s^\pm_{ext}$ and the various definitions of section 3.1, imply that

$$\alpha_{\phi^\pm} W_{s^\pm}[Y^\pm] \; := \; W_{(\phi^\pm)(s^\pm_{ext})|_{[0,2\pi]}}[Y^\pm]. \tag{67}$$

It is straightforward to see (either explicitly from equation (62) or abstractly using the fact that the periodicity of $\phi^\pm, Y^\pm, s^\pm_{ext}$ implies that one is effectively restricting attention to diffeomorphisms, graphs, charge networks and holonomies on S^1) that

$$\alpha(s^\pm, s'^\pm) = \alpha(\phi^\pm(s^\pm_{ext})|_{[0,2\pi]}, \phi^\pm(s'^\pm_{ext})|_{[0,2\pi]}. \tag{68}$$

Equations (65) and (68) imply that the Hamiltonian flow of (67) induces an automorphism of the Weyl algebra of holonomies. Note also that equation (63) is invariant under the action of this automorphism. This directly implies that group of finite gauge transformations is unitarily represented

[3] While our notation uses charge network labels, the operators $\hat{W}(s^\pm)$ and states $W(s^\pm)$ only depend on the equivalence classes of labels. See also Footnote 2 in this regard.

in the quantum theory. Let these unitary operators be denoted, as in the embedding sector, by $\hat{U}^{\pm}(\phi^{\pm})$. Their explicit action on the charge network basis can be defined from the GNS construction to be

$$\hat{U}^{\pm}(\phi^{\pm})W(s^{\pm}) \;:=\; W((\phi^{\pm})(s^{\pm}_{ext})|_{[0,2\pi]}). \tag{69}$$

3.4 The kinematic Hilbert space.

The kinematic Hilbert space $\mathcal{H}_{kin}$ is the product of the Hilbert spaces $\mathcal{H}^{\pm}_{kin}$ with

$$\mathcal{H}^{\pm}_{kin} \;=\; (\mathcal{H}^{\pm}_{E} \;\otimes\; \mathcal{H}^{\pm}_{M}) \tag{70}$$

so that

$$\mathcal{H}_{kin} \;=\; (\mathcal{H}^{+}_{E} \;\otimes\; \mathcal{H}^{+}_{M}) \;\otimes\; (\mathcal{H}^{-}_{E} \;\otimes\; \mathcal{H}^{-}_{M}). \tag{71}$$

$\mathcal{H}^{\pm}_{kin}$ is spanned by an orthonormal basis of equivalence classes of charge network states of the form $T_{s^{\pm}} \otimes W(s'^{\pm})$ with $s^{\pm} \;=\; \{\gamma(s^{\pm}), (k^{\pm}_{e^{\pm}_{1}}, ..., k^{\pm}_{e^{\pm}_{n^{\pm}}})\}$, $s'^{\pm} \;=\; \{\gamma(s'^{\pm}), (l^{\pm}_{e'^{\pm}_{1}}, ..., l^{\pm}_{e'^{\pm}_{m^{\pm}}})\}$.

The results of the previous subsections show that $\mathcal{H}_{kin}$ supports a *-representation of the *- algebras for the matter and embedding degrees of freedom, as well as a unitary representation of finite gauge transformations.

Consider, as above, the state $T_{s^{\pm}} \otimes W(s'^{\pm})$. The equivalence relation between charge networks is defined in Definition 1, section 3.1. Using this equivalence, it is straightforward to see that we can always choose $s^{\pm}, s'^{\pm}$ such that $\gamma(s^{\pm}) = \gamma(s'^{\pm})$. Then each edge $e^{\pm}$ of $\gamma(s^{\pm})$ is labelled by a pair of real charges $(k^{\pm}_{e}, l^{\pm}_{e})$. Note that such a choice graph and charge pairs is still not unique. However it is easy to see that a unique choice can be made if we require that the pairs of charges, $(k^{\pm}_{e^{\pm}}, l^{\pm}_{e^{\pm}})$, are such that no two consecutive edges are labelled by the same pair of charges. We shall denote this unique labelling by $\mathbf{s}^{\pm}$ so that

$$\mathbf{s}^{\pm} := \{\gamma(\mathbf{s}^{\pm}), (k^{\pm}_{e^{\pm}_{1}}, l^{\pm}_{e^{\pm}_{1}}), ..., (k^{\pm}_{e^{\pm}_{n^{\pm}}}, l^{\pm}_{e^{\pm}_{n^{\pm}}})\}, \tag{72}$$

with

$$k_{e^{\pm}_{I^{\pm}}} \neq k_{e^{\pm}_{(I+1)^{\pm}}} \text{ or/and } l_{e^{\pm}_{I^{\pm}}} \neq l_{e^{\pm}_{(I+1)^{\pm}}}. \tag{73}$$

The corresponding charge network state is denoted by $|\mathbf{s}^{\pm}\rangle$ so that

$$|\mathbf{s}^{\pm}\rangle = T_{s^{\pm}} \otimes W(s'^{\pm}) \tag{74}$$

with $\mathbf{s}^{\pm}$ defined from $s^{\pm}, s'^{\pm}$ in the manner discussed above. It follows from (46) and (69) that $\hat{U}^{\pm}(\phi^{\pm})$ maps $|\mathbf{s}^{\pm}\rangle$ to a new charge network state. We denote the new (unique) charge network label by $\mathbf{s}^{\pm}_{\phi^{\pm}}$ so that

$$|\mathbf{s}^{\pm}_{\phi^{\pm}}\rangle := \hat{U}^{\pm}(\phi^{\pm})|\mathbf{s}^{\pm}\rangle. \tag{75}$$

4 Unitary representation of Dirac observables.

4.1 Exponentials of mode functions.

Whereas $a_{(\pm)n}$ (23) depend on $Y^{\pm}(x)$, the basic operators of quantum theory are the holonomies $\hat{W}(s^{\pm})$. As in LQG, the representation of the holonomy operators on $\mathcal{H}_{kin}$ is not regular enough to allow a definition of $\hat{Y}^{\pm}(x)$ via a "shrinking of edges" procedure [3]. For example, let $s^{\pm}(t)$ be a 1 parameter family of charge networks such that $\gamma(s^{\pm}(t))$ has non- vanishing unit charge on only one of its edges. Let this edge contain x and let its coordinate length be t. Whereas, classically, $Y^{\pm}(x) = \lim_{t\to 0}\frac{W(s^{\pm}(t))-1}{it}$, it is easy to check that, as in LQG, the corresponding operators are not weakly continuous in t and the limit cannot be defined on the charge network basis. This leads to a regularization dependence in the definition of $\hat{a}_{(\pm)n}$ [3]. However, as we show below, suitably defined exponential functions of $a_{(\pm)n}, a^{*}_{(\pm)n}$ *can* be promoted to quantum operators in a regularization/triangulation independent manner. Let q_n, p_n be the real and imaginary parts of $a_{(\pm)n}$ so that

$$\begin{aligned} q_{(\pm)n} &= \int_{S^1} Y^{\pm}(x)\cos(nX^{\pm}(x)), \\ p_{(\pm)n} &= \int_{S^1} Y^{\pm}(x)\sin(nX^{\pm}(x)), \end{aligned} \tag{76}$$

and consider the functions

$$\begin{aligned} e^{i\alpha q_{(\pm)n}} &= e^{i\alpha\int_{S^1} Y^{\pm}(x)\cos(nX^{\pm}(x))} \\ e^{i\beta p_{(\pm)m}} &= e^{i\beta\int_{S^1} Y^{\pm}(x)\sin(nX^{\pm}(x))} \end{aligned} \tag{77}$$

where $\alpha,\ \beta\ \in \mathbf{R}$. These functions can be promoted to quantum operators as follows.

Let $f(X^{\pm})$ be a smooth periodic *real* function of $X^{\pm}$. Then $O^{\pm}_{f} := \int_{S^1} Y^{\pm}(x) f(X^{\pm}(x))$ are functions on the phase space of PFT. Next, restrict attention to the embedding sector Hilbert space $\mathcal{H}^{\pm}_{E}$ and consider the operator valued (on $\mathcal{H}^{\pm}_{E}$) function on the matter phase space, $O^{\pm}_{\hat{f}} := \int_{S^1} Y^{\pm}(x) f(\hat{X}^{\pm}(x))$. Since charge network states are eigen states of the embedding operator, we have that

$$O^{\pm}_{\hat{f}} T_{s^{\pm}} = \Big(\sum_{i=1}^{n} f(\hbar k^{\pm}_{e^{\pm}_i}) \int_{e^{\pm}_i} Y^{\pm}(x)\Big) T_{s^{\pm}}, \tag{78}$$

where $s^{\pm} = \{\gamma(s^{\pm}), (k^{\pm}_{e^{\pm}_1}, .., k^{\pm}_{e^{\pm}_{n^{\pm}}})\}$ and that,

$$\begin{aligned} e^{iO^{\pm}_{\hat{f}}} T_{s^{\pm}} &= e^{\sum_{i=1}^{n} f(\hbar k^{\pm}_{e^{\pm}_i}) \int_{e^{\pm}_i} Y^{\pm}(x)} T_{s^{\pm}}, \\ &= W(s^{\pm}_{f})[Y^{\pm}] T_{s^{\pm}}, \end{aligned} \tag{79}$$

where $s^{\pm}_f := \{\gamma(s^{\pm}), (f(\hbar k^{\pm}_{e^{\pm}_1}), ..., f(\hbar k^{\pm}_{e^{\pm}_{n^{\pm}}}))\}$. Equation (79) implies that we can define the operators $\widehat{\exp iO^{\pm}_f}$ corresponding to the functions $\exp iO^{\pm}_f$ via their action on the charge network states $T_{s^{\pm}} \otimes W(s'^{\pm}) \in \mathcal{H}^{\pm}$:

$$(\widehat{\exp iO^{\pm}_f})T_{s^{\pm}} \otimes W(s'^{\pm}) := \hat{W}(s^{\pm}_f)T_{s^{\pm}} \otimes W(s'^{\pm}). \tag{80}$$

Clearly, this is a manifestly regularization/triangulation independent definition. Moreover, since $s^{\pm}_f$ is constructed from the embedding part of the charge network, and since f is periodic, it is straightforward to check that $\widehat{e^{iO^{\pm}_f}}$ commute with the unitary operators corresponding to finite gauge transformations. Hence $\widehat{e^{iO^{\pm}_f}}$ are Dirac observables in quantum theory. It is also easy to check that

$$(\widehat{\exp iO^{\pm}_f})^{\dagger} = (\widehat{\exp iO^{\pm}_f})^{-1} = (\widehat{\exp iO^{\pm}_{-f}}) \tag{81}$$

so that the classical reality conditions are implemented.

By setting f to be the appropriate cosine (sine) function times α (β), we obtain the operators corresponding to the functions in equation (77). Clearly, these operators ($\forall \alpha, \beta \in \mathbf{R}, n > 0$) form an over- complete set of Dirac observables.

4.2 Conformal Isometries.

Regularization dependence also manifests in attempts to promote the generators of conformal isometries, $\Pi^{\pm}[U^{\pm}]$ (see equation (26), to operators on $\mathcal{H}_{kin}$. Choosing exponentials of these observables only partially alleviates this problem since (unlike the case of $a_{(\pm)n}$) the resulting operator suffers from operator ordering problems stemming from the fact that $\{\Pi_{\pm}(x), U^{\pm}(X^{\pm}(x))\} \neq 0$. Therefore, we focus on the Hamiltonian flows corresponding to finite conformal isometries.

The action of the Hamiltonian flows (corresponding to conformal isometries), $\alpha_{\phi^{\pm}_c}$, on $(X^{\pm}(x), Y^{\pm}(x))$ has been detailed in section 2.3.4. It remains to specify their action on the embedding momenta, $\Pi_{\pm}(x)$. The information in this specification can equally well be seeded in the action of $\alpha_{\phi^{\pm}_c}$ on the Hamiltonian flows $\alpha_{\phi^{\pm}}$ corresponding to finite gauge transformations by virtue of the facts that (a) the constraints (10) are linear in the embedding momenta and (b) this linear dependence is invertible by virtue of the non- degeneracy condition **(iv)** of section 2.3.1. Thus $\alpha_{\phi^{\pm}_c}$ are completely specified through equations (31),(32),(33). Accordingly, we seek a unitary representation of $\alpha_{\phi^{\pm}_c}$ by operators $\hat{V}(\phi^{\pm}_c)$ such that $\hat{V}^{\pm}(\phi^{\pm}_c)$ act trivially on the matter sector, commute with the operators $\hat{U}^{+}(\phi^{+})$ and $\hat{U}^{-}(\phi^{-})$ which implement gauge transformations, and transform $\hat{X}^{\pm}(x)$ through

$$\hat{V}^{\pm}(\phi^{\pm}_c)\hat{X}^{\pm}(x)(\hat{V}^{\pm})^{\dagger}(\phi^{\pm}_c) = \phi^{\pm}_c(\hat{X}^{\pm}(x)), \tag{82}$$

while leaving $\hat{X}^{\mp}(x)$ invariant.

We define $\hat{V}^{\pm}(\phi_c^{\pm})$ to act trivially on the matter Hilbert spaces $\mathcal{H}_M^+, \mathcal{H}_M^-$ and on the $\mp$ embedding Hilbert space $\mathcal{H}_E^{\mp}$. The action of $\hat{V}^{\pm}(\phi_c^{\pm})$ on $\mathcal{H}_E^{\pm}$ is defined as follows. Let $s = \{\gamma(s)(k^{\pm}_{e_1^{\pm}}, ..., k^{\pm}_{e_n^{\pm}})\}$ be a charge network. Define the charge networks $\phi_c^+(s^+), \phi_c^-(s^-)$ by

$$\phi_c^{\pm}(s^{\pm}) := \{\gamma(s^{\pm}), (\phi_c^{\pm}(k^{\pm}_{e_1^{\pm}}), ..., \phi_c^{\pm}(k^{\pm}_{e_n^{\pm}}))\}. \tag{83}$$

Then the action of $\hat{V}(\phi_c^{\pm})$ on the charge network state $T_{s^{\pm}} \in \mathcal{H}_E^{\pm}$ is defined to be

$$\hat{V}^{\pm}[\phi_c^{\pm}]T_{s^{\pm}} \;=\; T_{(\phi_c^{\pm})^{-1}(s^{\pm})}. \tag{84}$$

To reiterate, in the notation (83) we have that
$(\phi_c^{\pm})^{-1}(s^{\pm}) = \{\gamma(s^{\pm}), ((\phi_c^{\pm})^{-1}(k^{\pm}_{e_1^{\pm}}), ..., (\phi_c^{\pm})^{-1}(k_{e_n^{\pm}}))\}$.

From equation (84), the invertibility of the functions $\phi_c^{\pm}$ (which follows from equation (29)) and the inner product (40), it follows that
$\langle \hat{V}^{\pm}[\phi_c^{\pm}]T_{s^{\pm}} | \hat{V}^{\pm}[\phi_c^{\pm}]T_{s'^{\pm}}\rangle = \langle T_{s^{\pm}} | T_{s'^{\pm}}\rangle \;\forall s^{\pm}, s'^{\pm}$, thus showing unitarity. It is also straightforward to check, using the quasiperiodicity of the functions $\phi_c^{\pm}$ (30), that $\hat{V}^{\pm}[\phi_c^{\pm}]$ commutes with $\hat{U}(\phi^{\pm})$. By definition $\hat{V}^{\pm}[\phi_c^{\pm}]$ commutes with $\hat{U}(\phi^{\mp})$ and with the matter holonomies. Finally, it is easy to check that equation (82) holds when applied on any charge network state. Thus, our definition of $\hat{V}^{\pm}[\phi_c^{\pm}]$ provides a satisfactory definition of conformal isometries in quantum theory.

Note also that equation (84) implies that

$$\hat{V}^{\pm}[\phi_{1c}^{\pm}]\hat{V}^{\pm}[\phi_{2c}^{\pm}] = \hat{V}^{\pm}[\phi_{2c}^{\pm} \circ \phi_{1c}^{\pm}], \tag{85}$$

so that our definition of $\hat{V}^{\pm}[\phi_c^{\pm}]$ implies an anomaly free representation (by right multiplication) of the group of conformal isometries.

5 Physical state space by Group Averaging.

Only gauge invariant states are physical so that physical states Ψ must satisfy the condition $\hat{U}^{\pm}(\phi^{\pm})\Psi = \Psi,\ \forall \phi^{\pm}$. A formal solution to this condition is to fix some $|\psi\rangle \in \mathcal{H}_{kin}$ and set $\Psi = \sum |\psi'\rangle$ where the sum is over all distinct $|\psi'\rangle$ which are gauge related to ψ. A mathematically precise implementation of this idea places the gauge invariant states in the dual representation (corresponding to a formal sum over bras rather than kets) and goes by the name of Group Averaging. The "Group" is that of gauge transformations and the "Averaging" corresponds to the construction of a gauge invariant state from a kinematical one by giving meaning to the formal sum over gauge related states. Specifically (for details see Reference [9]), the physical Hilbert space can be constructed if there exists an anti-linear map η from a dense subspace $\mathcal{D}$ of the kinematical Hilbert space $\mathcal{H}_{kin}$, to its algebraic

dual $\mathcal{D}^*$, subject to certain requirements. The algebraic dual of $\mathcal{D}$ is defined to be the space of linear mappings from $\mathcal{D}$ to the complex numbers. The requirements which η needs to satisfy are as follows. Let $\psi_1, \psi_2 \in \mathcal{D}$, let $\hat{A}$ be a Dirac observable of interest and let $\phi^{\pm}$ be a gauge transformation with $\hat{U}^{\pm}(\phi^{\pm})$ being its unitary implementation on $\mathcal{H}_{kin}$. Let $\eta(\psi_1) \in \mathcal{D}^*$ denote the image of ψ_1 by η and let $\eta(\psi_1)[\psi_2]$ denote the complex number obtained by the action of $\eta(\psi_1)$ on ψ_2. Then for all $\psi_1, \psi_2, \hat{A}, \phi$ we require that
(1) $\eta(\psi_1)[\psi_2] = \eta(\psi_1)[\hat{U}(\phi)\psi_2]$
(2) $\eta(\psi_1)[\psi_2] = (\eta(\psi_2)[\psi_1])^*$, $\eta(\psi_1)[\psi_1] \geq 0$.
(3) $\eta(\psi_1)[\hat{A}\psi_2] = \eta(\hat{A}^{\dagger}\psi_1)[\psi_2]$.

Here, we choose $\mathcal{D}$ to be the finite span of charge network states. Clearly due to the split of '+' and '−' structures, we may consider averaging maps $\eta^{\pm}$ on the dense sets $\mathcal{D}^{\pm} \subset \mathcal{H}^{\pm}_{kin}$ separately. Here $\mathcal{D}^{\pm}$ is the finite span of states of the form $|\mathbf{s}^{\pm}\rangle$ (see section 3.4 for the notation used here and below). Define the action of $\eta^{\pm}$ on $|\mathbf{s}^{\pm}\rangle$ as

$$\begin{aligned} \eta^{\pm}(|\mathbf{s}^{\pm}\rangle) &= \eta_{[\mathbf{s}^{\pm}]} \textstyle\sum_{\mathbf{s'}^{\pm} \in [\mathbf{s}^{\pm}]} < \mathbf{s'}^{\pm}| \\ &= \eta_{[\mathbf{s}^{\pm}]} \textstyle\sum_{\phi^{\pm} \in Diff^P_{[\mathbf{s}^{\pm}]}\mathbf{R}} < \mathbf{s}^{\pm}_{\phi^{\pm}}|, \end{aligned} \tag{86}$$

where $[\mathbf{s}^{\pm}] = \{\mathbf{s'}^{\pm} | \mathbf{s'}^{\pm} = \mathbf{s}^{\pm}_{\phi^{\pm}}$ for some $\phi^{\pm}\}$, $Diff^P_{[\mathbf{s}^{\pm}]}\mathbf{R}$ is a set of gauge transformations such that for each $\mathbf{s'}^{\pm} \in [\mathbf{s}^{\pm}]$ there is precisely one gauge transformation in the set which maps $\mathbf{s}^{\pm}$ to $\mathbf{s'}^{\pm}$ and $\eta_{[\mathbf{s}^{\pm}]}$ is a positive real number depending only on the gauge orbit $[\mathbf{s}^{\pm}]$. The right hand side of equation (86) inherits an action on states in $\mathcal{D}$ from that of each of its summands. Due to the inner product (40), (64), only a finite number of terms in the sum contribute so that $\eta^{\pm}(|\mathbf{s}^{\pm}\rangle)$ is indeed in $\mathcal{D}^*$. It is straightforward to see that $\eta^{\pm}$ satisfies the requirements **(1)**, **(2)** and that a positive definite inner product $<,>_{phys}$ on the space $\eta^{\pm}\mathcal{D}^{\pm}$ can be defined through

$$< \eta^{\pm}(|\mathbf{s}^{\pm}_1\rangle), \eta^{\pm}(|\mathbf{s}^{\pm}_2\rangle) >_{phys} = \eta^{\pm}(|\mathbf{s}^{\pm}_1\rangle)[|\mathbf{s}^{\pm}_2\rangle]. \tag{87}$$

If in addition, **(3)** is also satisfied by $\eta^{\pm}$ the group averaging technique guarantees that the above inner product automatically implements the adjointness conditions on the Dirac observables (which act by dual action on $\mathcal{D}^{\pm *}$) [4] of section 4, by virtue of the fact that these conditions are implemented on $\mathcal{H}_{kin}$.

In section 5.2 we use the requirement **(3)** to constrain the positive real numbers $\eta_{[\mathbf{s}^{\pm}]}$ and thus bring down the enormous ambiguity in the inner product (87). While the analysis can be done, in principle, for all of $\eta^{\pm}[\mathcal{D}^{\pm}]$, we shall, for simplicity, restrict attention to a certain subspace of $\mathcal{D}^{\pm}$ which

[4] Given $\Psi^{\pm} \in \mathcal{D}^{\pm *}$, $\psi^{\pm} \in \mathcal{D}^{\pm}$ and $\hat{A}_{\pm}$ such that $\hat{A}^{\dagger}_{\pm}\psi^{\pm} \in \mathcal{D}^{\pm}$, define $\hat{A}_{\pm}\Psi^{\pm}$ through $\hat{A}_{\pm}\Psi^{\pm}[\psi^{\pm}] := \Psi^{\pm}[\hat{A}^{\dagger}_{\pm}\psi^{\pm}]$. This is the dual action.

is left invariant by finite gauge transformations as well as the Dirac observables of section 4. In section 5.1 we define this 'superselected' subspace. Finally, in section 5.3 we display an irreducible representation of the operator algebra generated by the Dirac observables in conjunction with the gauge transformations.

5.1 The chosen subspace of $\mathcal{D}$.

Consider the charge network state $T_{s^\pm} \otimes W_{s'^\pm}$. Let $\gamma(s^\pm)$ have $n^\pm$ edges and the let the embedding charges on these edges be such that:
(a) $\pm k^\pm_{e^\pm_{I^\pm}} \geq \pm k^\pm_{e^\pm_{(I^\pm-1)}} \quad I^\pm = 2,..,n^\pm$.
(b) $\pm(k^\pm_{e^\pm_{n^\pm}} - k^\pm_{e^\pm_1}) \leq \frac{2\pi}{\hbar}$.

These conditions are physically motivated. Condition **(a)** is the quantum analog of the classical non-degeneracy condition **(iv)** of section 2.3.1. Condition **(b)** (together with **(a)**) is the quantum version of the classical property (implicit in the smoothness of $X^\pm(x)$ in conjunction with **(ii), (iv)**) that the X circle wraps around the x circle once and only once.

Henceforth we shall restrict attention to charge network states subject to **(a)** and **(b)**. Note that these conditions define a superselection sector of $\mathcal{D}$ with respect to gauge transformations as well as the observables of section 4. We will refer to this subspace $\mathcal{D}_{(\mathbf{a})(\mathbf{b})}$.

5.2 Commutativity of $\eta^\pm$ with Dirac observables.

We focus on the '+' case and suppress the '+' superscripts wherever possible. The '−' case follows analogously. We aim to restrict $\eta_{[\mathbf{s}]}$ by subjecting it to condition **(3)** above. We choose $\hat{A} := \widehat{e^{iO^+_f}}$ (recall, from section 4.1, that $O^+_f := \int_{S^1} Y^+(x) f(X^+(x)))$. Thus we require that $\forall$ $\mathbf{s}$,

$$e^{i\int \widehat{Y^+ f(X^+)}}\eta(|\mathbf{s}\rangle) \;=\; \eta(e^{i\int \widehat{Y^+ f(X^+)}}|\mathbf{s}\rangle) \;. \tag{88}$$

As in equation (74) we set $|\mathbf{s}^\pm\rangle = T_{s^\pm} \otimes W(s'^\pm)$. The equivalence relation between charge network labels allows us, without loss of generality, to choose $\gamma(\mathbf{s}) = \gamma(s) = \gamma(s')$. Equations (80), (65), (62) imply that

$$e^{i\int \widehat{Y^+ f(X^+)}}|\mathbf{s}\rangle = \hat{W}_{s_f}|\mathbf{s}\rangle := e^{-\frac{i\hbar\alpha(s_f,s)}{2}}|\mathbf{s}(f)\rangle \tag{89}$$

where

$$\begin{aligned}
\mathbf{s} &= \{\gamma(\mathbf{s}), ((k_{e_1}, l_{e_1}), .., (k_{e_n}, l_{e_n}))\} && (90)\\
s &= \{\gamma(\mathbf{s}), (k_{e_1}, .., k_{e_n})\} && (91)\\
s_f &= \{\gamma(\mathbf{s}), (f(\hbar k_{e_1}), .., f(\hbar k_{e_n}))\} && (92)\\
\mathbf{s}(f) &= \{\gamma(\mathbf{s}), ((k_{e_1}, l_{e_1} + f(\hbar k_{e_1}), .., (k_{e_n}, l_{e_n} + f(\hbar k_{e_n})\} && (93)
\end{aligned}$$

$$\alpha(s_f, s) \;=\; \sum_{I=1}^{n} f(\hbar k_{e_I})[l_{e_{I+1}} \;-\; l_{e_{I-1}}],\; e_0 := e_n,\; e_{n+1} := e_1 \tag{94}$$

Recall (see section 3.4) that $\mathbf{s}$ denotes the unique labelling such that no two consecutive edges of $\gamma(\mathbf{s})$ have the same pair of charges. It is straightforward to see from equation (94) that for $I = 1, .., n-1$,

$$\begin{aligned} & k_{e_I} \neq k_{e_{I+1}} \quad \text{or/and} \quad l_{e_I} \neq l_{e_{I+1}} \\ \Rightarrow\; & k_{e_I} \neq k_{e_{I+1}} \quad \text{or/and} \quad l_{e_I} + f(\hbar k_{e_I}) \neq l_{e_{I+1}} + f(\hbar k_{e_{I+1}}). \end{aligned} \tag{95}$$

Thus, consistent with the use of bold face notation (see section 3.4), $\mathbf{s}(f)$ is also the unique labelling such that no two consecutive edges of its underlying graph (also chosen to be $\gamma(\mathbf{s})$) have the same pair of charges.

From footnote 4 (89), (68), the fact that $e^{i\int \widehat{Y^+ f(X^+)}}$ commutes with gauge transformations, and(86), it follows that the right hand side of (88) is

$$e^{i\int \widehat{Y^+ f}(X^+)}\eta(|\mathbf{s}\rangle) = \eta_{[\mathbf{s}]} e^{\frac{i\hbar\alpha(s_f,s)}{2}} \sum_{\phi \in Diff^P_{[\mathbf{s}]}\mathbf{R}} < \mathbf{s}(\mathbf{f})_\phi|. \tag{96}$$

and that the left hand side of (88) is

$$\eta(e^{i\int \widehat{Y^+ f}(X^+)}|\mathbf{s}\rangle) \;= \eta_{[\mathbf{s}(f)]} e^{\frac{i\hbar\alpha(s_f,s)}{2}} \sum_{\phi \in Diff^P_{[\mathbf{s}(f)]}\mathbf{R}} < \mathbf{s}(f)_\phi| \tag{97}$$

where $|\mathbf{s}(f)_\phi\rangle := \hat{U}(\phi)|\mathbf{s}(f)\rangle$. Thus we need to impose

$$\eta_{[\mathbf{s}]} \sum_{\phi \in Diff^P_{[\mathbf{s}]}\mathbf{R}} < \mathbf{s}(\mathbf{f})_\phi| = \eta_{[\mathbf{s}(f)]} \sum_{\phi \in Diff^P_{[\mathbf{s}(f)]}\mathbf{R}} < \mathbf{s}(f)_\phi| \tag{98}$$

It is easy to see that we may choose

$$Diff^P_{[\mathbf{s}]}\mathbf{R} = Diff^P_{[\mathbf{s}(f)]}\mathbf{R}. \tag{99}$$

This immediately follows from the fact that

$$\hat{U}(\phi)e^{\widehat{iO^+_f}}|\mathbf{s}\rangle \neq e^{\widehat{iO^+_f}}|\mathbf{s}\rangle \;\text{ iff }\; \hat{U}(\phi)|\mathbf{s}\rangle \neq |\mathbf{s}\rangle. \tag{100}$$

Equation (100) follows, in turn, from the invertibility of $e^{\widehat{iO^+_f}}$ (81) and its commutativity with $\hat{U}(\phi)$. Equations (99), (98) imply that

$$\eta_{[\mathbf{s}]} = \eta_{[\mathbf{s}(f)]}. \tag{101}$$

Next, we analyse the consequences of the restriction (101). There are 2 cases:
Case 1: $[\mathbf{s}]$ is such that there exists some $\mathbf{s} \in [\mathbf{s}]$, $\mathbf{s} = \{\gamma(\mathbf{s}), ((k_{e_1} l_{e_1}), .., (k_{e_n}, l_{e_n}))\}$ with

$$k_{e_1} < k_{e_2} < .... < k_{e_n}, \;\; (k_{e_n} - k_{e_1}) < 2\pi. \tag{102}$$

Case 2: The complement of Case 1.

We have analysed both cases. The analysis for Case 2 is quite involved and, in the interests of pedagogy, we do not present it here. We shall focus only on Case 1 in this paper. Accordingly, consider $\mathbf{s}$ as in Case 1. We define $\tilde{\mathbf{s}}$ to be the *embedding* charge network label which is obtained by dropping the matter charge labels from $\mathbf{s}$ so that $\gamma(\tilde{\mathbf{s}}) = \gamma(\mathbf{s})$ with the edges of $\gamma(\tilde{\mathbf{s}})$ carrying the same embedding charges as in $\mathbf{s}$. Since $\mathbf{s}, \mathbf{s}(f)$ have the same embedding charges and the same underlying graph, we could equally well have obtained $\tilde{\mathbf{s}}$ by dropping the matter charge labels from $\mathbf{s}(f)$. Thus, using the '$\tilde{}$' notation, we have that

$$\tilde{\mathbf{s}} = \tilde{\mathbf{s}}(f) = (\gamma(\mathbf{s}), (k_{e_1}, .., k_{e_n})). \tag{103}$$

Next, note that we can always choose f such that $f(\hbar k_{e_I}) = -l_{e_I}$, $I = 1, .., n$ so that $\mathbf{s}(f)$ has vanishing matter charges. Clearly the property that all matter charges vanish is a gauge invariant statement. This fact together with equation (103) implies that the set $[\mathbf{s}(f)]$ (with f chosen as above) is isomorphic to the set of *embedding* charge networks which are gauge equivalent to $\tilde{\mathbf{s}}$. Denoting the latter set by $[\tilde{\mathbf{s}}]$ we have, from equation (101) that $\eta_{[\mathbf{s}]}$ can only depend on the set $[\tilde{\mathbf{s}}]$. We denote this dependence through the notation

$$\eta_{[\tilde{\mathbf{s}}]} := \eta_{[\mathbf{s}]}. \tag{104}$$

An identical analysis holds for the conformal isometry operators $\hat{V}(\phi_c)$. Equation (84) implies that

$$\hat{V}(\phi_c)|\mathbf{s}\rangle =: |\phi_c^{-1}(\mathbf{s})\rangle. \tag{105}$$

$\mathbf{s}$ is given by equations (90), (102) and

$$\phi_c^{-1}(\mathbf{s}) = \{\gamma(\mathbf{s}), ((\phi_c^{-1}(k_{e_1}), l_{e_1}), .., (\phi_c^{-1}(k_{e_n}), l_{e_n}))\}. \tag{106}$$

The invertibility of ϕ_c and its periodicity imply that $\phi_c^{-1}(\mathbf{s})$ is the unique labelling such that no 2 consecutive edges have the same pairs of charges, and that the condition (102) is preserved by the action of $\hat{V}(\phi_c)$.

Condition **(3)** implies that, in obvious notation,

$$\eta_{[\mathbf{s}]} \sum_{\phi \in Diff^P_{[\mathbf{s}]}\mathbf{R}} < \phi_c^{-1}(\mathbf{s})|\hat{U}^\dagger(\phi) = \eta_{[\phi_c^{-1}(\mathbf{s})]} \sum_{\phi \in Diff^P_{[\phi_c^{-1}(\mathbf{s})]}\mathbf{R}} < \phi_c^{-1}(\mathbf{s})|\hat{U}^\dagger(\phi). \tag{107}$$

An argument identical to that in (100) implies that $Diff^P_{[\mathbf{s}]}\mathbf{R} = Diff^P_{[\phi_c^{-1}(\mathbf{s})]}\mathbf{R}$ so that

$$\eta_{[\mathbf{s}]} = \eta_{[\phi_c^{-1}(\mathbf{s})]}. \tag{108}$$

Clearly, given any pair of charge networks $\mathbf{s}_1, \mathbf{s}_2$ as in Case 1, with $\gamma(\mathbf{s}_1) = \gamma(\mathbf{s}_2)$ and with identical matter charges, there exists some ϕ_c such that

$|\mathbf{s}_2\rangle = \hat{V}(\phi_c)|\mathbf{s}_1\rangle$. This, in conjunction with equations (108), (104) implies that $\eta_{[\mathbf{s}]}$ can only depend on the set of graphs $[\gamma(\mathbf{s})]$ which are obtained by the action of gauge transformations on $\gamma(\mathbf{s})$. Specifically,

$$\begin{aligned} [\gamma(\mathbf{s})] &= \{\gamma' \text{ s.t. } \exists\phi \text{ s.t. } \gamma'_{ext} = \phi(\gamma_{ext})\} \\ \gamma &:= \gamma(\mathbf{s}), \end{aligned} \tag{109}$$

where we have used the notation defined in section 3.1. We denote this dependence of $\eta_{[\mathbf{s}]}$ through the notation

$$\eta_{[\mathbf{s}]} = \eta_{[\gamma(\mathbf{s})]} \tag{110}$$

This completes our analysis of the rigging map.

5.3 Cyclic representation

We focus on the '+' sector of the algebra of operators and the '+' sector of the state space. As in section 5.2 we suppress '+' superscripts. The analysis for the '−' case follows analogously. Cyclicity is defined with respect to an algebra of operators. Here the putative generators of the algebra are the Dirac observables of section 4 and the finite gauge transformations. As we shall see in section 6, neither does the commutator of two of the observables of section 4.1 yield a representation of the corresponding Poisson brackets nor does their product yield a representation of the appropriate Weyl algebra. As shown in section 6, the connection with classical theory is state dependent and only holds for semiclassical states (this is roughly similar to what happens for area operators in LQG [27]). Given this situation, we define the operator algebra in terms of the concrete representation on $\mathcal{H}_{kin}$ (or $\mathcal{H}_{phys}$) of the relevant operators rather than in terms of abstract representations of classical structures.

Since the operators of section 4 as well as those for finite gauge transformations are unitary (and hence bounded), the finite span of their products is well defined on $\mathcal{H}_{kin}$ so that it is possible to define the algebra of operators generated by these elementary ones in terms of the action of elements of this algebra on $\mathcal{H}_{kin}$. We denote this algebra of operators as $\mathcal{A}^{kin}_{D,G}$. In a similar manner, consider the algebra of operators generated by the action of the Dirac observables of section 4 on $\mathcal{H}_{phys}$. Denote this algebra by $\mathcal{A}_D$.

Fix a graph γ. Let $\mathbf{s}_\gamma$ be the set of charge networks such that $\forall\mathbf{s} \in \mathbf{s}_\gamma, \gamma(\mathbf{s}) = \gamma$ and $\mathbf{s}$ satisfies condition (102) on its embedding charges. Let $[\mathbf{s}_\gamma]$ be the set of charge networks which are gauge related to elements of $\mathbf{s}_\gamma$ i.e. $\forall\mathbf{s}' \in [\mathbf{s}_\gamma]\exists$ some gauge transformation ϕ and some $\mathbf{s} \in \mathbf{s}_\gamma$ such that $\mathbf{s}' = \mathbf{s}_\phi$. Finally, let $\mathcal{H}_{[\gamma]}$ be the (Cauchy completion of the) finite span $\mathcal{D}_{[\gamma]}$ ($\subset \mathcal{D}_{(\mathbf{a})(\mathbf{b})}$) of charge network states $|\mathbf{s}'\rangle$, $\mathbf{s}' \in [\mathbf{s}_\gamma]$.

The analysis of the preceding section shows that:
(1) $\mathcal{H}_{[\gamma]} \subset \mathcal{H}_{kin}$ provides a cyclic representation of the algebra $\mathcal{A}^{kin}_{D,G}$. Any

charge network state in $\mathcal{H}_{[\gamma]}$ is a cyclic state.
(2) Group averaging of states in $\mathcal{D}_{[\gamma]}$ yields a cyclic representation of the algebra $\mathcal{A}_D^{phys}$ i.e. $\mathcal{A}_D^{phys}$ is represented cyclically on $\mathcal{H}_{[\gamma],phys} \subset \mathcal{H}_{phys}$ where $\mathcal{H}_{[\gamma],phys}$ is the Cauchy completion (in the physical inner product) of $\eta(\mathcal{D}_{[\gamma]})$. The group average of any charge network state in $\mathcal{D}_{[\gamma]}$ is a cyclic state.

Note that both $\mathcal{H}_{[\gamma]}$ and $\mathcal{H}_{[\gamma],phys}$ are *non- separable.*

6 Semiclassical Issues.

An exhaustive analysis of semiclassical states is outside the scope of this paper. Instead, we focus on two issues related to semiclassicality. In section 6.1 we show that semiclassical states must be based on suitably defined 'weaves'. In section 6.2 we show that semiclassicality can be exhibited with respect to, at most, a countable number of the mode function operators of section 4.1.

6.1 Semiclassicality and Weaves.

Recall that in LQG, states which exhibit semiclassical behaviour for spatial geometry operators are based on graphs called weaves [28]. Here the (flat) spacetime geometry is encoded in the behaviour of the $\hat{X}^{\pm}(x)$ operators. Hence we define the notion of a weave as follows. The embedding charge network $s^{\pm} = \{\gamma(s^{\pm}), (k^{\pm}_{e^{\pm}_1}, .., k^{\pm}_{e^{\pm}_{N^{\pm}}})\}$ will be called a *weave* iff the embedding charges satisfy **(a)**,**(b)** of section 5.1 together with $k^{\pm}_{e^{\pm}_N} - k^{\pm}_{e^{\pm}_1} \approx \pm 2\pi$ and iff $N >> 1$. This is, of course, not a precise definition since $k^{\pm}_{e^{\pm}_N} - k^{\pm}_{e^{\pm}_1} \approx 2\pi$ and $N >> 1$. are not precise statements. Nevertheless this 'working' definition will suffice for our purposes.

Let $\psi^{\pm} \in \mathcal{H}^{\pm}_{kin}$ exhibit semiclassicality with respect to the $\pm$ sector observables of section 4.1. Further, let $\psi^{\pm}$ be an eigen state of $\hat{X}^{\pm}(x)$ (we shall relax this assumption later) so that $\psi^{\pm} = T_{s^{\pm}} \otimes \psi^{\pm}_M$, $\psi^{\pm}_M \in \mathcal{H}^{\pm}_M$. The analysis below is for the +-sector and can be trivially extended to the −-sector. In what follows we suppress the + superscript. From equation (80) it follows straightforwardly that

$$\langle [\widehat{e^{i\alpha q_m}}, \widehat{e^{i\alpha p_m}}] \rangle = -2i \sin(\frac{\alpha\beta\hbar}{2} f_{s,m}) \langle \widehat{e^{i\alpha q_m + i\beta p_m}} \rangle, \tag{111}$$

where

$$f_{s,m} := \sum_{I=1} \cos(\hbar m k_{e_I})(\sin(\hbar m k_{e_{I+1}}) - \sin(\hbar m k_{e_I})). \tag{112}$$

where $k_{e_{N+1}} := k_{e_1}$. In order to write (112) in a more useful form, we define the following:

$$\Delta k_{e_I} \quad := \quad k_{e_{I+1}} - k_{e_I}, \quad I = 1, .., N-1 \tag{113}$$

$$\Delta k_{e_N} \quad := \quad k_{e_1} - k_{e_N} + \frac{2\pi}{\hbar}. \tag{114}$$

Rearranging terms in (112) and using standard trigonometric identities we obtain that

$$f_{s,m} = \sum_{I=1}^{N} \sin(\hbar m \Delta k_{e_I}). \tag{115}$$

Since ψ is semiclassical we assume that, for some classical data (q_m, p_m),

$$\langle \widehat{e^{i\alpha q_m + i\beta p_m}} \rangle \approx e^{i\alpha q_m + i\beta p_m}, \tag{116}$$

and we require that as $\hbar \to 0$

$$\langle [\widehat{e^{i\alpha q_m}}, \widehat{e^{i\alpha p_m}}] \rangle \to i\hbar \{e^{i\alpha q_m}, e^{i\beta p_m}\} \tag{117}$$

where the Poisson bracket evaluates to

$$\{e^{i\alpha q_m}, e^{i\beta p_m}\} = -\alpha\beta 2\pi m e^{i\alpha q_m + i\beta p_m}. \tag{118}$$

Equations (111)- (118) imply that to leading order in $\hbar$

$$f_{s,m=1} \approx 2\pi. \tag{119}$$

Note that the eigen values of the embedding operators are in terms of

$$k_I := \hbar k_{e_I} \tag{120}$$

so that in the $\hbar \to 0$ (classical) limit, k_I does not vanish (except when $k_{e_I} = 0$). Hence, we investigate the conditions imposed on s by the requirement

$$|2\pi - \sum_{I=1}^{N} \sin(\Delta k_I)| < \epsilon, \quad \epsilon << 1. \tag{121}$$

where, similar to (113) we have defined

$$\Delta k_I \quad := \quad k_{I+1} - k_I, \quad I = 1, .., N-1 \tag{122}$$

$$\Delta k_N \quad := \quad k_1 - k_N + 2\pi. \tag{123}$$

Note that conditions $(\mathbf{a}), (\mathbf{b})$ of section 5.1 imply that

$$\Delta k_I \geq 0, \quad \sum_{I=1}^{N} \Delta k_I = 2\pi. \tag{124}$$

Intuitively, since $|\frac{\sin x}{x}| \leq 1$ and $= 1$ at $x = 0$, equations (121), (124) lead us to expect that $\Delta k_I, I = 1, .., N$ should be small. That this is indeed the case is shown in Lemmas 1- 3 in the Appendix. Clearly, the fact that $\Delta k_I \to 0$ as $\epsilon \to 0$ (see Appendix)

implies that s is a weave. Thus, we have shown that any kinematic semiclassical state which is an eigen state of the embedding operators must be based on a weave.

Next, consider an arbitrary kinematic state $|\psi\rangle = \sum a_i|s_i > \otimes|\psi_{iM}\rangle$ where a_i are complex coefficients, $|s_i\rangle$ are an orthonormal set of embedding charge network states and $|\psi_{iM}\rangle \in \mathcal{H}_M$. In order that this state satisfies equation (117), it turns out that $|\psi\rangle$ must be peaked around s_i such that s_i are weaves. This is shown in Lemma 4 of the Appendix.

Finally, consider an arbitrary physical state. Such a state is a linear combination of averages over embedding eigen states. Lemma 5 shows that such a state is peaked around averages of embedding eigen states which are based on weaves.

6.2 Semiclassicality and mode function operators: a no- go result.

We show that no states exist which are semiclassical with respect to the uncountable set of operators $\{\widehat{e^{i\alpha q_m}}, \widehat{e^{i\beta p_m}}, |\alpha - \alpha_0| < \epsilon, |\beta - \beta_0| < \delta\}$ for any fixed m, α_0, β_0 and any $\epsilon, \delta > 0$. First, consider states $|\psi\rangle$ which are embedding eigen states so that $|\psi\rangle = |s\rangle \otimes |\psi_M\rangle$. Here s is an embedding charge network and $|\psi_M\rangle \in \mathcal{H}_M$ can expanded as $|\psi_M\rangle = \sum_r b_r|s'_r\rangle$ where $\{|s'_r\rangle\}$ is a countable set of orthonormal matter charge networks.

The operators $\widehat{e^{i\alpha q_m}}, \widehat{e^{i\beta p_m}}$ act by changing the matter charge labels by sines and cosines of (m times) the embedding charges (see (80). Consider the set L of all matter charges on $s_r \forall r$ and construct the set ΔL of differences between all pairs of elements of L i.e. $\Delta L := \{l - l' \forall l, l' \in L\}$. Let $k_e, e \subset \gamma(s)$ be such that $\cos m\hbar k_e \neq 0$. Then, in any neighbourhood of α_0 we can choose uncountably many α such that $\alpha \cos m\hbar k_e \notin \Delta L$. Clearly for such α we have that $\langle\widehat{e^{i\alpha q_m}}\rangle = 0$. If $\cos m\hbar k_e = 0$ we can repeat the same argument with $\sin m\hbar k_e$ and conclude that $\langle\widehat{e^{i\beta p_m}}\rangle = 0$ for uncountable many β near β_0. Clearly, such behaviour is far from semiclassical. This argument can be suitably generalised for arbitrary states in $\mathcal{H}_{kin}$ as well as in $\mathcal{H}_{phys}$. The relevant material is in Lemma 6 and Lemma 7 of Appendix B.

7 Two open issues and their resolution.

Before we conclude this paper, a couple of points remain which we have not addressed as yet. First, it still remains to enforce **(ii)**, section 2.3.1 in order to ensure that the spatial topology is a circle. Second, we need to take care of the zero modes by imposing equation (25) in quantum theory and show that the results of section 6 continue to hold after this is done. We address these points in sections 7.1 and 7.2 below.

7.1 Identifying 2π shifted embeddings

Although the spatial inertial co-ordinate X ranges over $(-\infty, \infty)$, we need to identify $X \sim X + 2\pi$ in accordance with the discussion in section 2.3.1. Condition **(ii)**, section 2.3.1 states that two embeddings (X_1, T_1), (X_2, T_2) are equivalent if the following conditions are satisfied:

$$\begin{array}{l} X_1^+(x) = X_2^+(x) + 2m\pi \;\forall\; x \in [0, 2\pi], \\ X_1^-(x) = X_2^-(x) - 2m\pi \;\forall\; x \in [0, 2\pi]. \end{array} \tag{125}$$

We now show that this equivalence has already been taken care of at the physical state-space level. Let

$$\begin{aligned} \mathbf{s}^+ &= \{\ \gamma(\mathbf{s}^+), (k^+_{e_1^+}, ..., k^+_{e_N^+}), (l^+_{e_1^+}, ..., l^+_{e_N^+})\ \} \\ \mathbf{s}^- &= \{\ \gamma(\mathbf{s}^-), (k^-_{e_1^-}, ..., k^-_{e_M^-}), (l^-_{e_1^-}, ..., l^-_{e_M^-})\} \end{aligned} \tag{126}$$

The identification (126) in the classical theory implies the following equivalence condition in quantum theory:

$$|\mathbf{s}^+\rangle \otimes |\mathbf{s}^-\rangle \sim\ |\mathbf{s}^+_{2\pi m}\rangle \otimes |\mathbf{s}^-_{2\pi m}\rangle \tag{127}$$

where,

$$\begin{aligned} \mathbf{s}^+_{2\pi m} &= \{\ \gamma(\mathbf{s}^+), (k^+_{e_1^+} + 2m\pi, ..., k^+_{e_N^+} + 2m\pi), (l^+_{e_1^+}, ..., l^+_{e_N^+})\ \}, \\ \mathbf{s}^-_{-2\pi m} &= \{\ \gamma(\mathbf{s}^-), (k^-_{e_1^-} - 2m\pi, ..., k^-_{e_M^-} - 2m\pi), (l^-_{e_1^-}, ..., l^-_{e_M^-})\ \}. \end{aligned} \tag{128}$$

Next, note that for any integer m, there exist gauge transformations $\phi^\pm_{(m)}$ such that $\phi^\pm_{(m)} \cdot \mathbf{s}^\pm = \{\gamma(\mathbf{s}^\pm),\ (k^\pm_{e_1^\pm} \pm 2m\pi, ..., k^\pm_{e_N^\pm} \pm 2m\pi),\ (l^\pm_{e_1^\pm}, ..., l^\pm_{e_N^\pm})\}$. Thus $|\mathbf{s}^\pm\rangle$ and $|\mathbf{s}^\pm_{\pm 2\pi m}\rangle$ are gauge related so that

$$\eta^\pm(|\mathbf{s}^\pm\rangle)\ =\ \eta^\pm(|\mathbf{s}^\pm_{\pm 2\pi m}\rangle), \tag{129}$$

$$\Rightarrow \eta^+(|\mathbf{s}^+\rangle) \otimes\ \eta^-(|\mathbf{s}^-\rangle)\ =\ \eta^+(|\mathbf{s}^+_{2\pi m}\rangle) \otimes\ \eta^-(|\mathbf{s}^-_{-2\pi m}\rangle). \tag{130}$$

Equation (130) shows that the identification of 2π-shifted embeddings is *subsumed* by the identification of embeddings related by gauge transformations.

7.2 Taking care of the zero mode in quantum theory.

In section 7.2.1 we impose the condition $p = 0$ (see equation (25)) by appropriate group averaging. In section 7.2.2 we show that this does not alter the conclusions of section 6.

7.2.1 Imposition of $p = 0$ by averaging.

The conditions $\int_{S^1} Y^{\pm} = 0$ of equation (25) are equivalent to the conditions $e^{i\lambda^{\pm}\int_{S^1} Y^{\pm}} = 1,\ \forall\lambda^{\pm}$. The latter can be imposed by group averaging with respect to the operators $e^{i\widehat{\lambda^{\pm}\int_{S^1}} Y^{\pm}}$. Let $s^{\pm}_{\lambda^{\pm}}$ be matter charge networks with a single edge $e^{\pm} = [0, 2\pi]$ labelled by the charge $\lambda^{\pm}$ i.e. $s^{\pm}_{\lambda^{\pm}} = \{\gamma(s^{\pm}_{\lambda^{\pm}}) = [0, 2\pi], l^{\pm}_{e^{\pm}} = \lambda^{\pm}\}$. Clearly, we have that $e^{i\widehat{\lambda^{\pm}\int_{S^1}} Y^{\pm}} = \hat{W}(s^{\pm}_{\lambda^{\pm}})$.

It is easy to see that $U^{\pm}(\phi^{\pm})\hat{W}(s^{\pm}_{\lambda^{\pm}})U^{\pm}((\phi^{\pm})^{-1}) \;=\; \hat{W}(s^{\pm}_{\lambda^{\pm}})$. Thus we can average over the transformations generated by the zero-mode constraint before or after averaging over the group of gauge transformations. Since we have already averaged over the Virasoro group, we solve the zero-mode constraint by defining a Rigging map $\overline{\eta}^{\pm} : \eta^{\pm}(\mathcal{D}^{\pm}_{(a)(b)}) \to \eta^{\pm}(\mathcal{D}^{\pm}_{(a)(b)})^{*}$. Recall that $\mathcal{D}^{\pm}_{(a)(b)}$ (defined in section 5.1) is the finite span of charge networks subject to the conditions $(\mathbf{a}), (\mathbf{b})$ of section 5.1. $\eta^{\pm}(\mathcal{D}^{\pm}_{(a)(b)})^{*}$ is the algebraic dual of $\eta^{\pm}(\mathcal{D}^{\pm}_{(a)(b)})$. Before defining $\overline{\eta}^{\pm}$, note that,

$$\hat{W}(s^{\pm}_{\lambda^{\pm}})|\mathbf{s}^{\pm}\rangle \;=:\; |\mathbf{s}^{\pm}_{\lambda^{\pm}}\rangle \tag{131}$$

where $\mathbf{s}^{\pm}_{\lambda^{\pm}}$ is obtained from $\mathbf{s}^{\pm} = \{\gamma(\mathbf{s})^{\pm}, \vec{k}^{\pm}, \vec{l}^{\pm}\}$ by adding $\lambda^{\pm}$ to all the matter charges. We now define,

$$\overline{\eta}^{\pm}(\eta^{\pm}(|\mathbf{s}^{\pm}\rangle)) = \overline{\eta}_{[[\mathbf{s}^{\pm}]]_0}\eta_{[\mathbf{s}^{\pm}]}(\textstyle\bigoplus_{\lambda^{\pm}\in\mathbf{R}}\sum_{\phi^{\pm}\in Diff^{P}_{[\gamma(\mathbf{s}^{\pm})]}\mathbf{R}}\langle(\mathbf{s}^{\pm}_{\phi^{\pm}})_{\lambda^{\pm}}| \tag{132}$$

The equivalence class $[[\mathbf{s}^{\pm}]]_0$ is defined via following relation.
$[\mathbf{s}^{\pm}] \sim [\mathbf{s}^{\pm}_1]$ iff for any $\{\gamma(\mathbf{s}^{\pm}), \vec{k}^{\pm}, \vec{l}^{\pm}\} \in [\mathbf{s}^{\pm}]$, $\exists$ $(\{\gamma(\mathbf{s}^{\pm}), \vec{k}^{\pm}, \vec{l}^{\pm} + \lambda_{\pm}\} \in [\mathbf{s}^{\pm}_1]$ for some $\lambda_{\pm} \in \mathbf{R}$.

Once again the ambiguity in the rigging map contained in $\overline{\eta}_{[[\mathbf{s}^{\pm}]]_0}$ can be reduced by demanding that $\overline{\eta}^{\pm}$ commutes with the observables. It can be checked that for the super-selected sector of $\mathcal{H}_{phy}$ defined in section 5.2, we have $\overline{\eta}_{[[\mathbf{s}^{\pm}]]_0} \;=\; \overline{\eta}_{[\gamma(\mathbf{s}^{\pm})]}$. Setting $\tilde{\eta}_{[\gamma(\mathbf{s}^{\pm})]} := \overline{\eta}_{[\gamma(\mathbf{s}^{\pm})]}\eta_{[\gamma(\mathbf{s}^{\pm})]}$, we have that the inner product on $\overline{\eta}^{\pm}(\mathcal{D}^{\pm\ ss}_{phy})$ is given by,

$$\langle\overline{\eta}^{\pm}(\eta^{\pm}(|\mathbf{s}^{\pm}\rangle)|\overline{\eta}^{\pm}(\eta^{\pm}(|\mathbf{s}^{\pm}_1\rangle)\rangle \;=\; \tilde{\eta}_{[\gamma(\mathbf{s}^{\pm})]}\bigoplus_{\lambda^{\pm}}\Big(\ \eta^{\pm}(|\mathbf{s}^{\pm}\rangle)[|\mathbf{s}_{1,\lambda^{\pm}}\rangle]\Big), \tag{133}$$

7.2.2 Semiclassical Issues.

Since the zero mode operator $\hat{W}(s^{\pm}_{\lambda^{\pm}})$ leaves the embedding part of the states in $\mathcal{H}_{kin}$ and $\mathcal{H}_{phys}$ untouched, it is easy to see that the proofs of section 6.1 and appendix A still apply after the zero mode averaging is done. Thus, semiclassical states which satisfy the $p = 0$ constraint are necessarily based on weaves.

It is also straightforward to see that the results of section 6.2 apply after zero mode group averaging. While the line of argument is roughly similar

to that in section 6.2 and appendix B, there are some differences. In the interests of brevity, we provide only a skeleton of the argument below. As usual we shall suppress the $\pm$ superscripts.

The averaging with respect to $\overline{\eta}$ slightly complicates matters because there is an additional sum over matter charge networks wherein matter charges associated with charge network states are all incremented by the same amount. As a result, it is necessary to consider pairs of edges subject to conditions on their embedding charges. This is in contrast to the role of single edges (with cosines or sines of ($\hbar$ times) their embedding charges being non- vanishing) in the arguments of section 6.1 and appendix B. Specifically, consider a state decomposition defined in terms of embedding charge networks s_j as in equations (138) and (151). Separate the values taken by the index j into a set C_1 and its complement, C_2, where $j \in C_1$ iff for fixed m, there exist a pair of edges $e_I(j), e_J(j) \in \gamma(s_j)$ such that $\cos m\hbar k_{e_I(j)} \neq \cos m\hbar k_{e_J(j)}$.

Next, with a slight abuse of notation, for each $j \in C_1$ fix a pair of edges $e_I(j), e_J(j) \in \gamma(s_j)$ such that $\cos m\hbar k_{e_I(j)} \neq \cos m\hbar k_{e_J(j)}$. As in appendix B, define ΔL to be the set of differences of all matter charges which occur in the expansions (138), (151), (157). Also define $\Delta^2 L$ to be the set of all differences between pairs of elements of ΔL. For each $j \in C_1$ define $\Delta^2 L_j$ to be the set of elements obtained by dividing each element of $\Delta^2 L$ by $\cos m\hbar k_{e_I(j)} - \cos m\hbar k_{e_J(j)}$. Let $\Delta^2 L_{C_1} := \cup_{j \in C_1} \Delta^2 L_j$. The set $\Delta^2 L_{C_1}$ is countable so that there are uncountably many α in any neighbourhood of α_0 such that $\alpha \notin \Delta^2 L_{C_1}$. It can then be checked that $\langle \widehat{e^{i\alpha q_m}} \rangle$ obtains contributions only from terms labelled by $j \in C_2$.

Finally, we show that such terms are of negligible measure. Note that for $j \in C_2$ we have that $\cos m\hbar k_{e_I(j)} = \cos m\hbar k_{e_J(j)}$ for any pair of edges $e_I(j), e_J(j) \in \gamma(s_j)$. It is then straightforward to see that for such j, the function $f_{s_j,m}$ (defined by equations (144), (112)) vanishes identically. Then the arguments of section 6.1 and appendix A imply that the contribution from $j \in C_2$ must be negligible for semiclassicality to hold.

Similar arguments can be made for $\langle \widehat{e^{i\beta p_m}} \rangle$ by replacing cosines with sines in the above argument.

8 Discussion of results and open issues.

In this work, we constructed a quantization of PFT similar to that used in LQG. Quantum states are in correspondence with graphs (i.e. collections of edges) in the spatial manifold. The edges of these graphs are labelled by a set of real valued embedding and matter charges. These charge network states are analogs of the spin network states in LQG. There, however, the labels are integer valued. Such a labelling is also, in principle, possible here. Had the holonomies of section 3 been based on charge networks with

embedding charges which were integer multiples of $\frac{2\pi}{L}$ for some fixed integer L and matter charges which were also integer multiples of some appropriate dimensionful unit, such holonomies would still separate points in phase space by virtue of the fact that they were based on arbitrary graphs (this is similar to what happens in LQG). Such a choice would lead to states with integer valued charges. However it is not clear if (a large enough subset of) the Dirac observables of section 4 preserve the space spanned by these integer-charge network states. It would be useful to investigate this issue in detail.

The polymer quantization of the embedding variables replaces the classical (flat) spacetime continuum with a discrete structure consisting of a countable set of points. This can be seen as follows. The canonical data $X^{\pm}(x)$ is a map from S^1 into the flat spacetime $(S^1 \times R, \eta)$ and embeds the former into the latter as a spatial Cauchy slice. Any gauge transformation generated by the constraints maps this data to new embedding data which, in turn, define a new Cauchy slice in the flat spacetime. In particular, the action of the one parameter family of gauge transformations generated by smearing the constraints with some choice of "lapse-shift" type functions N^A (see section 2) generates a foliation of $(S^1 \times R, \eta)$. Consider the image set in $(S^1 \times R, \eta)$ of the set of all embeddings which are gauge related to a given one. From the above discussion it follows that this image set is exactly the flat spacetime $(S^1 \times R, \eta)$ itself. Next, consider the corresponding quantum structures. Any charge network state is an eigen state of $\hat{X}^{\pm}(x)$. Consider a charge network state, $|\mathbf{s}^+\rangle \otimes |\mathbf{s}^-\rangle$ with $|\mathbf{s}^{\pm}\rangle = T_{s^{\pm}} \otimes W_{s'^{\pm}}$, where $s^{\pm}$ satisfy the conditions $(\mathbf{a}), (\mathbf{b})$ of section 5.1. From equation (42)- (44) it follows that the set of eigen values $\lambda_{x,s^{\pm}}$ for all $x \in [0, 2\pi]$ describes a finite set of points on a spacelike Cauchy surface in $(S^1 \times R, \eta)$. These points have light cone coordinates $(X^+, X^-) = (\lambda_{x,s^+}, \lambda_{x,s^-})$. The action of any gauge transformation on such a charge network state yields another charge network state whose eigen values lie, once again, on a Cauchy slice in $(S^1 \times R, \eta)$. From equation (46) it follows that the set of eigen values for all possible gauge related charge network states is countable and defines a corresponding set of points in $(S^1 \times R, \eta)$. The gauge invariant state obtained by group averaging a charge network state is a sum over all distinct gauge equivalent states and hence contains the elements of this discrete structure. The discrete structure is a good approximant of the continuum spacetime $(S^1 \times R, \eta)$ for charge networks with a large number of embedding charges i.e. for weave states. Thus, it is not surprising that semiclassicality requires states to be based on weaves as in section 6.1 and appendix A.

In contrast to the embedding charges, the matter charges do not have a direct physical interpretation because charge network states are not eigen states of the matter holonomies. As a tentative, provisional interpretation we choose to think of them, rather imprecisely, as measuring excitations of the matter. Since, on the constraint surface, the classical data $(X^{\pm}(x), Y^{\pm}(x))$ correspond to free scalar field data $Y^{\pm}(x)$ on the slice

$(X^+(x), X^-(x))$ in flat spacetime, we interpret a charge network state $|\mathbf{s}^+\rangle \otimes |\mathbf{s}^-\rangle \in \mathcal{H}_{kin}$ as specifying excitations of matter on the discretized "quantum" slice specified by the embedding charges. The action of a gauge transformation on a charge network state can then be interpreted as evolving the matter excitations on the 'initial' quantum slice specified by this state to the new one specified by the gauge related charge network state. Since the physical state obtained as the group average of a charge network state contains all distinct gauge related states, it follows that such a physical state may be interpreted, roughly, as a "history". It may be useful to attempt an interpretation of physical states in LQG along these lines.

An over- complete set of Dirac observables corresponding to exponential functions of the standard annihilation- creation modes of free scalar field theory are represented as (unitary) operators in the polymer representation. Note that in contrast to the assumption of Reference [9], here the commutator between two such operators does not close as in the case of Weyl algebras. Indeed, as shown in section 6.1, the commutator only approximates the corresponding Poisson bracket for semiclassical states based on weaves. This underlines the fact that in a general covariant theory involving spacetime geometry, classical structures are typically not approximated in the $\hbar \to 0$ limit unless it is possible to coarse grain/smoothen away the underlying discreteness of the quantum spacetime. Nevertheless the action of the basic Dirac observables is well defined and there is no obstruction to the quantization procedure.

The results of section 6.2 imply that semiclassical analysis requires a choice of a countable subset of these observables. One possibility is to choose, for each n, a pair $\alpha, \beta << \frac{1}{\sqrt{\hbar}}$ and define the approximants to $\hat{q}_n, \hat{p}_n$ by $\frac{\widehat{e^{i\alpha q_n}} - \widehat{e^{-i\alpha q_n}}}{2i\alpha}$, $\frac{\widehat{e^{i\beta p_n}} - \widehat{e^{-i\beta p_n}}}{2i\beta}$. However, there is no natural choice of α, β and so, while the quantization constructed in this paper is free of the "triangularization" choices which occur in the definition of the quantum dynamics of LQG, an element of choice does appear when semiclassical issues are confronted. Note, however, that the results of section 6.1 indicate that any physical semiclassical state necessarily has an associated (gauge invariant) structure, namely that of a weave. [5] The "spacing" of the weave (i.e. $\hbar\Delta k_I$ of section 6.1 and the Appendix A) provides a natural scale for α, β. Thus, our viewpoint is that since choices of Dirac observables can be tied (however tenuously) to structures already present in the semiclassical states, ambiguities (if present) in definitions of the quantum dynamics are more worrying because quantum dynamics is defined for all states, not only semiclassical ones.

The above discussion naturally brings us to the efficacy of polymer PFT

[5]Note that in contract to the weaves of Reference [28] which approximate a spatial geometry, here it is the (flat) *spacetime* geometry which is being approximated by virtue of the discussion in the second paragraph of this section.

as a toy model for LQG. We believe that the quantization provided here is a useful testing ground for proposed definitions of quantum dynamics in canonical LQG. It would be of interest to construct the quantum dynamics of the model along the lines of Reference [17] and compare the resulting physical Hilbert space with the one considered here. Proposals for examining semiclassical issues [20, 21] may also be tested here. One of the outstanding problems in LQG [23, 29] is the relation between states in LQG and the Fock states of perturbative gravity. Since PFT admits a Fock quantization [1, 2] equivalent to the standard flat spacetime free scalar field Fock representation, one may enquire as to how Fock states arise from the polymer Hilbert space. Since the results of section 6.2 suggest that the operators corresponding to exponentials of mode functions do not possess the requisite continuity for the annihilation- creation modes themselves to be defined as operators, it is difficult to identify Fock states in terms of their properties with respect to the action of the annihilation- creation operators. However, as a first step, it may be possible to identify candidate states corresponding to the Fock vacuum by using the Poincare invariance of the latter. Specifically, since the operators corresponding to finite Poincare transformations are available (as a subset of the conformal isometry operators of section 4), one could try and group average with respect to these operators.

Another open issue pertains to the representation appropriate to the case of non- compact spatial topology. The quantization here explicitly incorporates the compact spatial topology S^1. Here, the unit of length has been chosen so that the circumference of the T = constant circle is 2π. By allowing the circle to have an arbitrarily large circumference, it may be possible to transit to polymer PFT on $R \times R$ and compare the resulting quantization with the Infinite Tensor Product proposal of Thiemann ([30, 31]).

Appendix

A. Lemmas concerning Semiclassicality and Weaves.

Lemma 1: If $\Delta k_J \geq \pi$ (see (120),(122)) for some J, $1 \leq J \leq N$ then $-1 \leq f_{s,m=1} \leq \pi$.
Proof: Let $\Delta k_J \geq \pi$. Equations (124) imply that

$$\sum_{I \neq J} \Delta k_I \leq \pi, \tag{134}$$

and, hence, that

$$\Delta k_I|_{I \neq J} \leq \pi. \tag{135}$$

This in conjunction with the fact that $|\frac{\sin x}{x}| \leq 1$ implies that

$$\sum_{I=1}^{N} \sin \Delta k_I \leq \sum_{I \neq J} \Delta k_I + \sin \Delta k_J \leq \pi. \tag{136}$$

From equation (135) and $\Delta k_J \geq \pi$, we have that

$$\sum_{I=1}^{N} \sin \Delta k_I \geq -1. \tag{137}$$

The Lemma follows immediately from equations (136), (137) and the definition (115) of $f_{s,m=1}$
Lemma 2: If $\Delta k_I \leq \pi,\ I = 1,..,N$ (see (120),(122)) then $0 \leq f_{s,m=1} \leq 2\pi$.
Proof: This follows immediately from the fact that $|\frac{\sin x}{x}| \leq 1$ in conjunction with equations (124) and the definition (115) of $f_{s,m=1}$.
Lemma 3: Equation (121) implies that as $\epsilon \to 0,\ \Delta k_I \to 0,\ I = 1,..,N$ and $N \to \infty$.
Proof:

From Lemma 1 and equation (121) it follows that for sufficiently small ϵ, it must be the case that $\Delta k_I \leq \pi, I = 1,..,N$.

Next, let α be the minimum value of the bounded, continuous function $\frac{\sin\theta}{\theta}$ in the interval $[0, \frac{\pi}{2}]$ (here $\frac{\sin\theta}{\theta}|_{\theta=0} := 1$). Define the function $f(x) := x - \sin x - \frac{\alpha}{6}x^3$. It is easy to check that $\frac{df}{dx} \geq 0,\ x \in [0,\pi]$ and that $f(x=0) = 0$. This implies that $x - \sin x \geq \frac{\alpha}{6}x^3,\ x \in [0,\pi]$. This in conjunction with equations (124), (121) implies that $\sum_{I=1}^{N}(\Delta k_I)^3 < \frac{6\epsilon}{\alpha}$ so that $\Delta k_I \to 0,\ I = 1,..,N$ as $\epsilon \to 0$. This in turn, together with (124), implies that $N \to \infty$ as $\epsilon \to 0$.
Lemma 4: Any normalised $|\psi\rangle \in \mathcal{H}_{kin}$ admits the expansion:

$$\begin{aligned}
|\psi\rangle &= \sum_j a_j |s_j, \psi_{jM}\rangle,\ \ |s_j, \psi_{jM}\rangle := |s_j\rangle \otimes |\psi_{jM}\rangle, && (138)\\
\langle s_i|s_j\rangle &= \delta ij,\ s_j = \{\gamma(s_j), (k_{e^j_1}, .., k_{e^j_{n_j}})\} && (139)\\
\langle \psi_{jM}|\psi_{jM}\rangle &= 1, && (140)\\
\sum_j |a_j|^2 &= 1. && (141)
\end{aligned}$$

Here s_j are embedding charge labels, $e^j_I, I = 1,..,n_j$ are the edges of the graph underlying s_j, a_j are complex coefficients and $|\psi_{jM}\rangle \in \mathcal{H}_M$.

If $|\psi\rangle$ is semiclassical then the coefficients a_j are such that $|\psi\rangle$ is peaked around s_j such that s_j are weaves.
Proof: The proof closely mirrors the arguments of section 6.1. Semiclassicality implies that to leading order in $\hbar$,

$$\langle\psi|[\widehat{e^{i\alpha q_m}}, \widehat{e^{i\alpha p_m}}]|\psi\rangle \approx i\hbar\{e^{i\alpha q_m}, e^{i\beta p_m}\} = -i\hbar\alpha\beta 2\pi m e^{i\alpha q_m + i\beta p_m} \tag{142}$$

Using equations (138), (80), (142), we have that

$$\sum_j |a_j|^2 2\sin(\frac{\alpha\beta\hbar}{2} f_{s_j,m})\langle s_j,\psi_{jM}|\widehat{e^{i\alpha q_m+i\beta p_m}}|s_j,\psi_{jM}\rangle \approx \hbar\alpha\beta 2\pi m e^{i\alpha q_m+i\beta p_m} \tag{143}$$

where

$$f_{s_j,m} = \sum_{I=1}^{n_j} \sin m\Delta k_I^j, \tag{144}$$

and $\Delta k_I^j := k_{I+1}^j - k_I^j$, $k_{n_j+1}^j := k_1^j$ and we have set $k_I^j := \hbar k_{e_I^j}$.

From Lemmas 1 and 2 it follows that

$$-1 \leq f_{s_j,m=1} \leq 2\pi. \tag{145}$$

Since $f_{s_j,m=1}$ is bounded, equation (143) implies that to leading order in $\hbar$, we have that

$$\sum_j |a_j|^2 f_{s_j,m=1}\langle s_j,\psi_{jM}|\widehat{e^{i\alpha q_1+i\beta p_1}}|s_j,\psi_{jM}\rangle e^{-i\alpha q_1-i\beta p_1} \approx 2\pi \tag{146}$$

Denote the left hand-side of equation (146) by LHS. Equation (146) implies that

$$|LHS - 2\pi| \leq \delta, \quad \delta << 1. \tag{147}$$

Taking absolute values of both sides of equation (146) and using (145), (141) and the fact that $\widehat{e^{i\alpha q_m+i\beta p_m}}$ is a bounded operator of norm 1, we have that

$$2\pi \geq \sum_j |a_j|^2 |f_{s_j,m=1}| \geq |LHS|. \tag{148}$$

From (148), (147) we have that $\delta \geq |2\pi - LHS| \geq 2\pi - |LHS| \geq 2\pi - \sum_j |a_j|^2|f_{s_j,m=1}|$, so that

$$\sum_j |a_j|^2 |f_{s_j,m=1}| \geq 2\pi - \delta. \tag{149}$$

Let $J_<$ be the set of all j such that $|f_{s_j,m=1}| \leq 2\pi - \delta^{\frac{1}{2}}$ and let $\sum_{j\in J_<} |a_j|^2 = P_<$. Then (145),(149) imply that $P_<(2\pi - \delta^{\frac{1}{2}}) + (1-P_<)2\pi \geq 2\pi - \delta$ so that $P_< \leq \delta^{\frac{1}{2}}$. Thus as $\delta \to 0$, almost all j are such that $|f_{s_j,m=1}| \geq 2\pi - \epsilon$ where we have set $\epsilon := \delta^{\frac{1}{2}}$. Using (145), this, in turn, implies that for small enough ϵ,

$$f_{s_j,m=1} \geq 2\pi - \epsilon. \tag{150}$$

This brings us back to equation (119) with $s = s_j, m = 1$. The analysis subsequent to that equation implies that such s_j must be a weave.

Lemma 5: Let $|\psi\rangle \in \mathcal{H}_{phys}$ be semiclassical. Then $|\psi\rangle$ is peaked at group averages of embedding eigen states which are based on weaves.

Proof: Recall that $|\psi\rangle$ is in the completion of $\eta(\mathcal{D})$ where $\mathcal{D}$ is the finite span of charge network states. It is then straightforward to see that any such $|\psi\rangle$ admits the expansion:

$$|\psi\rangle = \sum_j a_j \eta(|s_j\rangle \otimes |\psi_{jM}\rangle), \tag{151}$$

such that

$$\eta(|s_i\rangle \otimes |\psi_{iM}\rangle)[|s_j\rangle \otimes |\psi_{jM}\rangle] = \delta_{ij}, \tag{152}$$

and $|s_i\rangle$, $|s_j\rangle$ are not gauge related if $i \neq j$ i.e. for $i \neq j$ and $\forall \phi$,

$$|s_i\rangle \neq \hat{U}(\phi)|s_j\rangle. \tag{153}$$

Here s_j is an embedding charge network label, ϕ is a gauge transformation and $|\psi_{jM}\rangle \in \mathcal{H}_M$. We shall use the notation of Lemma 4 for the edges and charge labels of s_j. Note that $|\psi_{jM}\rangle$ is such that $\eta(|s_j\rangle \otimes |\psi_{jM}\rangle) \in \mathcal{H}_{phys}$ as implied by (152). Using (87), the normalization $< \psi|\psi >_{phys} = 1$ implies that

$$\sum_j |a_j|^2 = 1 \tag{154}$$

Semiclassicality implies that, to leading order in $\hbar$,

$$\langle\psi|[\widehat{e^{i\alpha q_m}}, \widehat{e^{i\alpha p_m}}]|\psi\rangle_{phys} \approx -i\hbar\alpha\beta 2\pi m e^{i\alpha q_m + i\beta p_m}, \tag{155}$$

where the '-' sign in the right hand side is due to the fact that operators act on $\mathcal{H}_{phys}$ by dual action (see Footnote 4). Using equations (80) and (153), we have that

$$\begin{aligned} & -\sum_j |a_j|^2 2i \sin(\frac{\alpha\beta\hbar}{2} f_{s_j,m}) \langle \eta(|s_j\rangle \otimes |\psi_{jM}\rangle),\ \widehat{e^{i\alpha q_m + i\beta p_m}} \eta(|s_j\rangle \otimes |\psi_{jM}\rangle)\rangle_{phys} \\ \approx \quad & -i\hbar\alpha\beta 2\pi m e^{i\alpha q_m + i\beta p_m}. \end{aligned} \tag{156}$$

Here $f_{s_j,m}$ is defined as in Lemma 4. [6] This is the analog of equation (143) of Lemma 4. The analysis of Lemma 4 subsequent to that equation applies here identically thus proving Lemma 5.

B. Lemmas concerning the no go result of section 6.2.

Lemma 6: No states $|\psi\rangle \in \mathcal{H}_{kin}$ exist which are semiclassical with respect to the uncountable set of operators $\{\widehat{e^{i\alpha q_m}}, \widehat{e^{i\beta p_m}}, |\alpha - \alpha_0| < \epsilon, |\beta - \beta_0| < \delta\}$ for any fixed m, α_0, β_0 and any $\epsilon, \delta > 0$.

[6] It is straightforward to check that $f(s_j, m)$ (144) is a *gauge invariant* function of s_j i.e. $f_{s_j,m} = f(s'_j, m) \forall s'_j$ such that $\exists$ a gauge transformation ϕ such that $|s'_j\rangle = \hat{U}(\phi)|s_j\rangle$.

Proof: As in Lemma 4 of Appendix A, any $|\psi\rangle \in \mathcal{H}_{kin}$ admits the expansion (138)- (141). Additionally we may expand $|\psi_{jM}\rangle$ in terms of matter charge networks so that for any fixed j,

$$|\psi_{jM}\rangle = \sum_{r^j} b_{r^j} |s'_{r^j}\rangle \tag{157}$$

$$\langle s'_{r_1^j} | s'_{r_2^j}\rangle = \delta_{r_1^j, r_2^j} \tag{158}$$

where r^j varies over a countable set (as, of course, does j), b_{r^j} are complex coefficients and and s'_{r^j} are matter charge networks.

Let C be the set of all j such that $\gamma(s_j)$ has at least one edge $e(j)$ with embedding charge $k_{e(j)}$ such that $\cos m\hbar k_{e(j)} \neq 0$. For every $j \in C$ choose an edge $e^j \subset \gamma(s_j)$ with embedding charge k_{e^j} such that

$$c_j := \cos m\hbar k_{e(j)} \neq 0. \tag{159}$$

Let S be the set of all j such that $j \notin C$. Clearly, for each $j \in S$ we can fix an edge $e^j \in \gamma(s_j)$ such that its charge label k_{e^j} satisfies

$$s_j := \sin m\hbar k_{e(j)} \neq 0. \tag{160}$$

Next, let L be the set of all matter charges which occur in $s'_{r^j} \forall j, r$. Let ΔL be the set of differences between all pairs of elements of L i.e. $\Delta L = \{l - l' \forall l, l' \in L\}$. For every $j_C \in C$, $j_S \in S$, define the sets $\Delta L_{j_C}, \Delta L_{j_S}$ whose elements are obtained by dividing elements of ΔL by c_{j_C}, s_{j_S} (see (159),(160)) i.e. $\Delta L_{j_C} := \{\frac{x}{c_{j_C}} \forall x \in \Delta L\}$, $\Delta L_{j_S} := \{\frac{x}{s_{j_S}} \forall x \in \Delta L\}$. Finally, let $\Delta L_C := \cup_{j_C \in C} L_{j_C}, \Delta L_S := \cup_{j_S \in S} L_{j_S}$.

Note that $\Delta L_C, \Delta L_S$ are both countable sets. It follows that in any neighbourhood of α_0, β_0 there exist uncountably many α, β such that $\alpha \notin \Delta L_C, \beta \notin \Delta L_S$. Then from (80) and the fact that $\widehat{e^{i\beta p_m}}$ is an operator of unit norm, it follows that for such α, β,

$$|\langle \psi \widehat{e^{i\alpha q_m}} |\psi\rangle| = \sum_{j \in S} |a_j|^2, \tag{161}$$

$$|\langle \psi \widehat{e^{i\beta p_m}} |\psi\rangle| \leq \sum_{j \in C} |a_j|^2 = 1 - \sum_{j \in S} |a_j|^2. \tag{162}$$

Semiclassicality requires that both (161) and (162) be close to unity. Clearly, this is not possible.

Lemma 7: No states $|\psi\rangle \in \mathcal{H}_{phys}$ exist which are semiclassical with respect to the uncountable set of operators $\{\widehat{e^{i\alpha q_m}}, \widehat{e^{i\beta p_m}}, |\alpha - \alpha_0| < \epsilon, |\beta - \beta_0| < \delta\}$ for any fixed m, α_0, β_0 and any $\epsilon, \delta > 0$.

Proof: As in Lemma 5, Appendix A, any $|\psi\rangle \in \mathcal{H}_{phys}$ admits the expansion (151)- (153). Further $|\psi_{jM}\rangle$ can be expanded as in equation (157)- (158) of Lemma 6. Note that the antilinearity of η implies that we may rewrite equation (151) as

$$|\psi\rangle = \eta(\sum_j a_j^* |s_j\rangle \otimes |\psi_{jM}\rangle). \tag{163}$$

Next, let us construct the sets $\Delta L_C, \Delta L_S$ (as defined in Lemma 6) for the state $\sum_j a_j^* |s_j\rangle \otimes |\psi_{jM}\rangle \in \mathcal{H}_{kin}$. It follows straightforwardly from the periodicity of the cosine and sine functions in conjunction with the action of gauge transformations (75) that we may choose the sets $\Delta L_C, \Delta L_S$ in such a way that they are identical for any (kinematic) state which is gauge related to the state $\sum_j a_j^* |s_j\rangle \otimes |\psi_{jM}\rangle$. Thus the sets $\Delta L_C, \Delta L_S$ can be chosen so as to depend only on the physical state $|\psi\rangle$, and it is straightforward to see that, as in Lemma 6, if we choose $\alpha \notin \Delta L_C$,$\beta \notin \Delta L_S$, we obtain equations (161), (162) with $|\psi\rangle$ as in (163). This proves the Lemma.

C. Choice of units.

In this appendix we summarize dimensions of various operators and parameters of the theory. We have set the speed of light c to be unity.

$$\begin{aligned}
&[S_0] = ML = [\hbar] \\
&[f] = M^{\frac{1}{2}} L^{\frac{1}{2}},\ [\pi_f] = M^{\frac{1}{2}} L^{-\frac{1}{2}} \\
&[X^{\pm}] = L, [\Pi_{\pm}] = ML^{-1} \\
&[q_{(\pm)n}] = M^{\frac{1}{2}} L^{-\frac{1}{2}} = [p_{(\pm)n}]
\end{aligned} \tag{164}$$

where $[n] = L^{-1}$.

The dimensions of the above fields naturally imply the dimensions of the various charges and parameters involved in the theory.

$$\begin{aligned}
&[k_e] = M^{-1},\ [l_e] = M^{-\frac{1}{2}} L^{-\frac{1}{2}} \\
&[\alpha] = M^{-\frac{1}{2}} L^{-\frac{1}{2}}
\end{aligned} \tag{165}$$

where the parameter α occurs in the exponentiated observables defined in (77).

Throughout this paper, we have fixed the units such that length of the $T=$ constant circle is 2π. Thus the only arbitrary scale in the theory is the mass scale.

References

[1] K. Kuchař, *Phys.Rev.***D39**, 2263 (1989).

[2] C. Torre and M. Varadarajan, *Phys.Rev.***D58**, 064007 (1998).

[3] A. Laddha, *Class.Quant.Grav.***24**, 4969 (2007); *Class.Quant.Grav.***24**, 4989 (2007).

[4] C. Rovelli, *Living Rev.Rel.***1:1** (1998).

[5] A. Ashtekar and J. Lewandowski, *Class.Quant.Grav.***21**, R53 (2004).

[6] L. Smolin, eprint:hep-th/0408048.

[7] T. Thiemann, *Lect.Notes Phys.***631**,41 (2003).

[8] L. Smolin and C. Rovelli, *Nucl.Phys.***B331**, 80 (1990).

[9] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao and T. Thiemann, *J.Math.Phys.***36**, 6456 (1995).

[10] C. Rovelli and L. Smolin, *Nucl.Phys.***B442**, 593 (1995); Erratum-ibid.**B456**, 753 (1995).

[11] A. Ashtekar and J. Lewandowski, *Class.Quant.Grav.***14**, A55 (1997).

[12] A. Ashtekar and J. Lewandowski, *Adv.Theor.Math.Phys.***1**, 388 (1998).

[13] C. Rovelli, *Phys.Rev.Lett.***77**, 3288 (1996).

[14] A. Ashtekar, J. Baez, K. Krasnov and A. Corichi, *Phys.Rev.Lett.***80**, 904 (1998).

[15] L. Lewandowski, A. Okolow, H. Sahlmann and T. Thiemann, *Commun.Math.Phys.***267**, 703 (2006).

[16] C. Fleischhack in *Quantum Gravity: Mathematical Models and Experimental Bou nds* edited by Fauser *et. al.*, Birkhauser Basel, 2007.

[17] K. Giesel and T. Thiemann, *Class.Quant.Grav.***24**, 2465 (2007).

[18] A. Perez, *Class.Quant.Grav.***20**, R43 (2003).

[19] M. Varadarajan and J. Zapata,*Class.Quant.Grav.***17**, 4085 (2000).

[20] T. Thiemann, *Class.Quant.Grav.***18**, 2025 (2001).

[21] A. Ashtekar and S. Fairhurst, *Class.Quant.Grav.***20**, 1031 (2003).

[22] D. Giulini and D. Marolf, *Class.Quant.Grav.* **16**, 2479 (1999).

[23] M. Varadarajan, *Class.Quant.Grav.* **22**, 1207 (2005).

[24] K. Kuchař, *Phys.Rev.* **D39**, 1579 (1989).

[25] See for example *Quantum Mechanics*, E. Merzbacher (John Wiley and Sons 1970).

[26] See for example *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*, R. W. Wald (The University of Chicago Press 1994).

[27] A. Ashtekar, A. Corichi and J. A. Zapata, *Class.Quant.Grav.* **15**, 2955 (1998).

[28] A. Ashtekar, C. Rovelli and L. Smolin, *Phys.Rev.Lett.* **69**, 237 (1992).

[29] Carlo Rovelli, *Phys.Rev.Lett.* **97**, 151301 (2006).

[30] T. Thiemann, O. Winkler , *Class.Quant.Grav.* **18**, 4997 (2001).

[31] H. Sahlmann, T. Thiemann, O. Winkler , *Nucl.Phys.B* **606**, 401 (2001).